Higher Secondary
IT MATHS LAB MANUAL
SECOND YEAR

2023

State Council of Educational Research and Training (SCERT) KERALA

Preface

IT Maths lab is introduced in higher secondary level to make the learning process more creative and child friendly. Maths Lab opens a platform for *learning by doing* and it accelerates learning by making it joyous. Mathematics skills are inherent in a child and it blossoms through schooling. Children acquire knowledge of numbers and numerical skills in their early childhood itself through plays and activities. The way of presentation of mathematical concepts becomes a bit abstract in higher classes and it spoils the genuine interest of children in the subject. For example, we attain the concept of area of a rectangle or volume of a rectangular prism through appropriate activities. But it becomes a herculean task to find an activity suitable for the class room, to impart the idea of maximum surface area of a rectangular prism whose volume is fixed. If a learning process demands tedious sequence of logical arguments, then it would be, better if we make it an activity and let the students do it and learn the concept.

Fortunately it has become easy today to makeover a concept to an activity by the use of technology and kids are specially equipped in keeping the pace with it. Software like GeoGebra contribute a lot in this makeover. Activities in IT Maths Lab are regulated with the help of GeoGebra. Detailed Lab Manuals for class 11 and 12 are also set for helping the students.

- In this manual 27 labs are given related to *class 12* topics.

- Each lab consists of 2 or 3 Activities and some additional activities.

- Additional Activities are optional, meant for only those who are interested.

- Manual is so set that students can do the activities independently using it.

- Step by step explanation is given for each Activity.

- Separate markings are given for those steps which are to be written in the observation book.

- Each student has to keep an observation book, which is to be examined and rectified by the teacher.

Our class room environment limits the scope of innovative learning. Hopefully this will be well overcome by the IT Maths lab

Acknowledgements

General education in Kerala is undergoing drastic changes by incorporating innovative methods in the teaching learning process. Different initiatives are taken after scientific research and planning, to enhance the academic excellence of the learner community. Such a new wave of thought has been bringing in many a change in the teaching of Mathematics in Kerala. At the primary level, Mathematics may be of concrete concepts, which, during its gradation to the higher secondary level, becomes a greater field of abstraction which requires proper visualisation. Learners get the opportunity to learn Maths using free softwares like GeoGebra up to Standard X. But at the higher secondary level, its continuity is lost to an extent as there is no scope for IT enabled learning in Mathematics. Considering this , the SCERT Kerala has introduced the concept of IT Maths Lab for higher secondary students from the academic year 2019-20 onwards. Also IT Maths lab manuals are prepared for the first and second year at the higher secondary level. The IT Maths Lab Manual of second year students will surely help all children as a source of inspiration in exploring concepts in Mathematics and will definitely support all learners to attain the concepts meaningfully. With regards,

Dr. Jayaprakash R K,
Director,
SCERT

,

The Team

Contents

Lab 17

Relations

Aim

- To explore the concept of reflexive, symmetric and transitive relations geometrically.

Concepts

- Reflexive relation, symmetric relation, transitive relation

Discussion

We discuss the geometry of reflexive, symmetric and transitive relations defined on the set $\mathbb{R}$ of real numbers.

Activity 17.1 Reflexive and Symmetric relations

Procedure

- Consider the relation R in the set $\mathbb{R}$ of real numbers, defined as
  R = $\{(x, y) : x \leq y^2\}$.

- Draw the region representing the relation using the input command `x<=y^2`

- Draw the line $y = x$

  📖 Observe the region and check whether the relation is reflexive. Write your argument.

- Plot a point A in the region and take its reflection A′ about the line $y = x$ (using `Reflect Object in Line` tool).
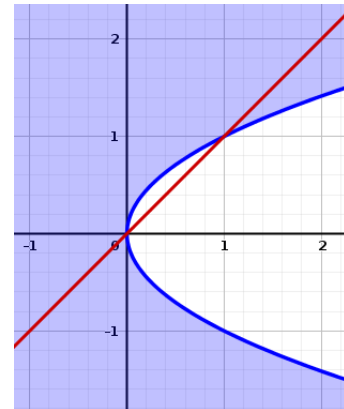
📖 Observe the coordinates of A and A′. How are they related?

📖 With the help of the points A and A′ say whether the relation is symmetric or not.

📖 How can we determine the symmetry of a relation by mere observation of the graph?

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Activity 17.2 Transitive relations

We construct an applet to check the transitivity of a relation.

Procedure

- Draw the line $y = x$

- Plot a point A and reflect it on above line to get the point A′

- Draw the line passing through A′ and parallel to the y axis.

- Plot any point B on above line

- Plot the point C using the input command (x(A),y(B))

- Complete the triangle ABC (not mandatory). Show the coordinates of the triangle.

📖 Observe the coordinates of the points A, B and C and write down their peculiarity.

📖 What do you infer about the transitivity of a relation R defined in the set of real numbers, if the points A and B are in the region represented by R and C is outside the region.

📖 Verifies the transitivity of the relation mentioned in Activity 17.1 based on above findings.
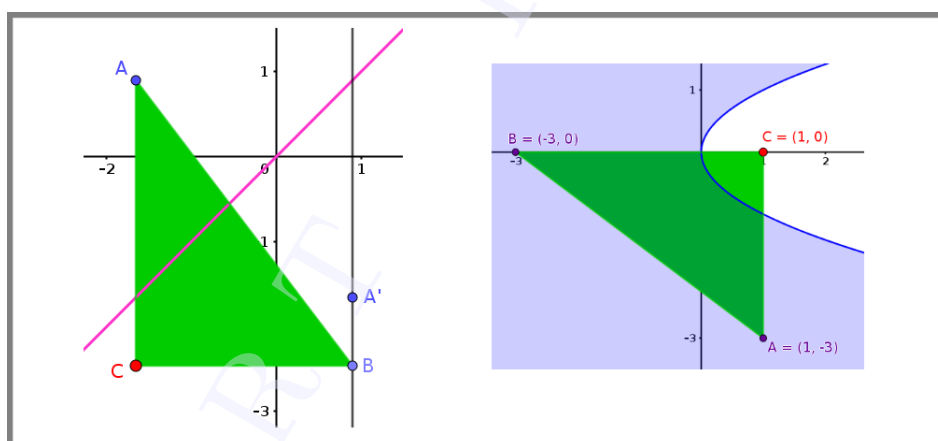
Activity 17.3 Transitive relations 2

In above activity the point C moves depending on A and B. In some problems, it is more convenient if we can move the point C independently. In this activity we create such an applet.

- Plot a point C

- Draw the line passing through C and parallel to the y-axis and take any point A on the line

- Plot the point B(y(A),y(C)).

- Complete the triangle ABC
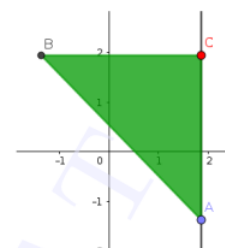
What do you infer about the transitivity of a relation R defined in the set of real numbers, if C is outside the region and the points A and B are in the region represented by R?.

Check whether the following relations defined on the set of real numbers $\mathbb{R}$ are reflexive, symmetric or transitive. Verify your answer geometrically.

i) $R_1 = \{(x, y) : y \leq x\}$

ii) $R_2 = \{(x, y) : y \leq -x\}$

iii) $R_3 = \{(x, y) : y \geq 2x\}$

iv) $R_4 = \{(x, y) : x^2 + y^2 \geq 9\}$

v) $R_5 = \{(x, y) : y \leq x^2\}$

vi) $R_6 = \{(x, y) : x \leq y^3\}$

Lab 18

Functions

Aim

- To visualise one to one and onto function geometrically

- To restrict the domain and co-domain of functions so as to make it a bijection

- To visualise composition of two functions geometrically

Concepts

- One to one and onto functions

- Composition of functions

Discussion

We discuss the properties of graphs of one to one and onto functions, how to determine whether a given function is one to one and onto geometrically, how to restrict the domain and co-domain of functions so as to make them bijections. We plot the graph of $g \circ f$ from the graphs of the functions f and g

Activity 18.1 One to one and Onto Functions

Procedure

- Draw the graph of $f(x) = x^2$
- Create a slider **a** and plot the point $(0, a)$
- Draw a line through the point and perpendicular to the y axis.

📖   If for any value of a, the line meets the graph of a function at more than one point, can we say whether the function one to one or not. Why ?

📖   If for all values of a, the line meets the graph of a function at atleast one point, can we say whether the function onto or not. Why ?

📖   Using above applet, say whether the function $f : \mathbb{R} \to \mathbb{R}$ defined above is one to one or onto.

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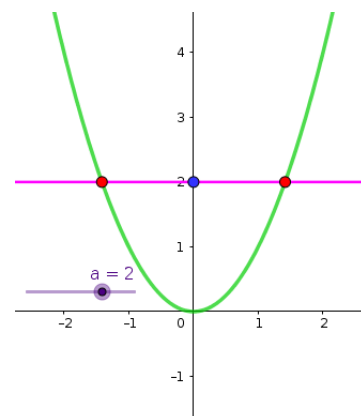
- Keeping above ideas in mind, one can say whether a function is one to one or onto, by observing its graph, even without drawing the horizontal line.

- Save the file as Activity 18.1

Activity 18.2 Domain and range of bijective functions

📖 Draw the graphs of the following functions defined from $\mathbb{R}$ to $\mathbb{R}$ and say whether they are one to one or onto by observing the graph of the functions. Find the range of the function in each case.

	Function	One to one	Onto	Range
i	$x^2 + 2$			
ii	$x^3 - 3x^2 + 3$			
iii	$3\sin(x)$			

📖 The following functions are defined from a subset of the set of real numbers. Say whether they are one to one or onto. Find their range.

i) $f : \mathbb{R} - \{3\} \to \mathbb{R} - \{1\}$ defined by $f(x) = \dfrac{x-2}{x-3}$

ii) $f : [-1, 1] \to \left[-1, \dfrac{1}{3}\right]$ defined by $f(x) = \dfrac{x}{x+2}$

iii) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \dfrac{x}{1+|x|}$

📖 What is the peculiarity of the range of an onto function ?

Activity 18.3 Bijective Functions

We can make a function one to one by restricting its domain and onto by restricting its co-domain. If we define a function to its range, it becomes onto. From the graph of a function we can easily select a suitable subset of the domain to make it one to one

Procedure

- 📖 Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 1$. Is it one to one or onto ?

- Draw the graph of above function

  📖 Redefine the function by restricting its co-domain so that it becomes onto

- From the graph of the function we can see that if we restrict the domain to $[0, \infty)$ the function becomes one to one.

- So the function $f : [0, \infty) \to [1, \infty)$ defined by $f(x) = x^2$ is a bijection.

- Draw the graph of above function. [Use input command `if(x>=0,x^2+1)` or `if(x>=0,f)` or `Function(f,0,infinity)`]

- If we define $f : (-\infty, 0] \to [1, \infty)$ then also it becomes a bijection. Draw the graph of this function.
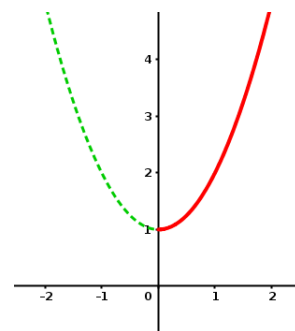
📖 Restrict domain and co-domain of the following functions so that they becomes bijections

	Function	Domain	Co-domain
i	$|x-2|$	$[2, \infty)$	
ii	$x^2 + 6x + 7$		
iii	$|\sin x|$		

📖 For each of above functions find one more domain which make them one to one. Draw the corresponding graphs.

Activity 18.4 Composition of functions

We create an applet to describe the composition of two functions geometrically

Procedure

- Draw the graph of $f(x) = x^2$ and $g(x) = \sin x$
- Draw the line $y = x$
- Create a slider **a**
- Plot the point $A(a, f(a))$
- Draw the line passing through A and parallel to x axis and plot its point of intersection B with the line $y = x$
- Draw the line passing through B and parallel to the y axis and plot its point of intersection C with the graph of the function g
- Draw the following lines and take their point of intersection D

  – Passing through A and parallel to the y axis

  – Passing through C and parallel to the x axis
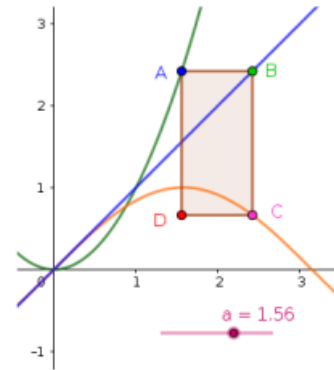
- Complete the rectangle $ABCD$ and hide the lines

  📖 Write the co ordinates of the points B,C and D in terms of **a**

  📖 Trace the point D and animate the slider. What does the graph represents? Use locus tool to get the clear path.

- Create input boxes for the functions f and g.

  📖 Following functions are of the form $f \circ g(x)$. Identify f and g and use above applet to trace the path of the given functions

i) $\sin |x|$ ii) e^{logx}

Additional Activities

Activity 18.A Composition - Physical situation

We discuss a physical situation in which the concept of composition of functions included.

Use Applet ML18.A

About the applet
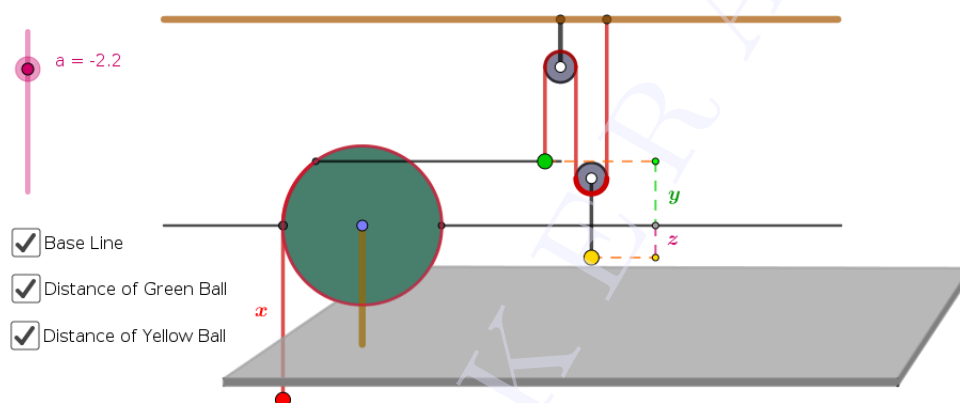
A rope is wound on a disc of radius 1 meter . A red ball is attached at the end of the rope. A green ball is attached to a point on the circumference of the disc by means of a rod which always remains horizontal. An yellow ball is attached to the green ball using two pulleys. We can wind/unwind the rope using the slider **a**. When **a**=0 all the balls lie on the base line, the horizontal line passing through the centre of the disc.

If we unwind the disc so that the red ball moves x meters downwards, let the green ball moves y meters and yellow ball moves z meters from the base line. We take the movement of a balls as positive, if it is above the base line and negative if it is below the base line.

📖 Write y as a function of x.

📖 Write z as a function of y.

📖 Write z as a function of x.

📖 Suppose we pull down the red ball at the rate of 2m/s. Write x, y and z as functions of the time t.

📖 Suppose that the red ball freely falls under gravitational force. Write x, y and z as functions of the time t.

Lab 19

Invertible Functions

Aim

• To explore invertible functions

Concepts

• Bijective functions

• Inverse of a function

• Composition of functions

Discussion

We discuss the method of finding the value of $f^{-1}(a)$, for a given real number a, from the graph of $f(x)$. Methods of drawing graph of inverse of a function is also discussed.

Activity 19.1 Inverse of a Function 1

If f is an invertible function and (x, y) is a point on the graph of f then $f(x) = y$ and $f^{-1}(y) = x$. We use this idea to find the value of $f^{-1}(a)$, for given values of a, from the graph of $f(x)$.

Procedure

• Draw the graph of $f(x) = x^2, x \geq 0$

• Create a slider **a**

• Plot the point $A(0, a)$

• Draw the line passing through A and parallel to the x axis

• Plot the point of intersection B of the above line with the curve

• Draw the line through B, perpendicular to the x axis

• Plot the point of intersection C with the x axis

• Hide the lines and draw the line segments AB and BC

• Show the coordinates of C
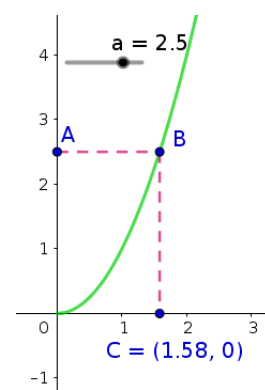
   📖 Using this applet, how can we find an approximate value of $\sqrt{a}$ for a given value of a ?

   📖 Find approximate values of the following

   i) $\sqrt{3}$ ii) $\sqrt{4.5}$

- Note : if we need more accuracy we can increase the number of decimal places. (Options →Rounding)

- Save the applet as Activity 19.1

Activity 19.2 Inverse of a Function 2

Procedure

Use the applet ML19.2

About the applet:

This applet is similar to that we created in the above activity. In addition there are two input boxes to edit the function f and the slider **a**

📖 Select suitable functions and find approximate values of the following

		Function	Value
i	$\sqrt[3]{4}$		
ii	$\frac{1}{\sqrt{5}}$		

Activity 19.3 Inverse of a Function 3

Procedure

- Find the inverse of the function $f(x) = x^3$

- Draw the graphs of $f(x)$ and $f^{-1}(x)$

- Draw the line $y = x$

  📖 Compare the graphs of f and f^{-1} with respect to the above line. What do you observe ?

  📖 If we know the graph of function f how can we draw a rough sketch of the graph of f^{-1} ?

- Using the tool *Reflect about line* we can draw the reflection of a graph of a function about a line (Click on the graph of the function and then on the line $y = x$)

- Draw the graph of function as $f(x) = x^2$ and find its reflection on the line $y = x$. Does it represent the graph of a function? Why?

- To get the graph of inverse of a function , restrict the domain of the function -if needed- in order to make it one to one, draw the graph of the function in the restricted domain and find the reflection on the line $y = x$

  📖 Find the inverse of the following functions algebraically and draw their graphs. Draw the graph of the given function and find its reflection on the line $y = x$. Verify whether it coincides with the graph of the inverse. Find the domain and range of the given function.

	Function	Inverse	Domain	Range
i	$\sqrt{x+2}$			
ii	$3x^2 - 12x + 15$			
iii	$\frac{x}{x+2}$			

👉

In algebra view, the equation of a reflected curve is shown only in its parametric form. The command Invert(f) gives the graph of inverse of a function and its equation in explicit form. It is not necessary to restrict the domain to get the inverse. But this command also has some drawbacks.

It works only if the function contains just one x. If f is a quadratic function, then the command : Invert(CompleteSquare(f)) gives the inverse

If f is a rational function with first degree polynomials in numerator, then the command Invert(PartialFractions(f)) gives the inverse

In CAS view " invert' command directly works even if the function contains more than one x

Activity 19.4 Composition and Inverse

Procedure

- Use the applet Activity 18.4

- Give $f(x) = 3x - 2$ and $g(x) = \dfrac{x+2}{3}$

  📖 Observe the graphs of $f \circ g$ and $g \circ f$. What do you see ? Why it happens so ?

- Find the inverse of the following functions and verify your answer using above applet.

	$f(x)$	$f^{-1}(x)$
i	$\frac{x-2}{3}$	
ii	$(3-x^3)^{\frac{1}{3}}$	

Additional Activities

Activity 19.A Self Invertible Functions

We know that the inverse of $f(x) = \dfrac{1}{x}$ is the function itself. We may call it as a self invertible function. Here we discuss a method to generate self invertible functions from existing one. The logic behind the construction is that the graphs of self invertible functions are symmetric with respect to the line $y = x$ (why ?)

Procedure

- draw the graph of the function $f(x) = \dfrac{1}{x}$ and the line $y = x$

- The graph is symmetric with respect to the line. If we shift it along the line, it keeps its symmetry

- Create a slider **a**

- Create the function $g(x) = f(x - a) + a$. Find the simplified form of g using the command `Simplify(g)`.

- Create an input box for f and save the applet as Activity 19.A

  📖 We get different self invertible functions for different values of a. Find some functions

and verify it using the applet Activity 18.4

📖 Starting from the functions $\dfrac{2}{x}, \dfrac{-3}{2x}, \dfrac{2x-5}{2x-2}$ generate more self invertible functions.

Activity 19.B Composition Machine

Last year we have studied functions as input - output machines, which gives outputs corresponding to the given inputs. Here we compare composition of functions with combination of machines in which output of one of the machine is the input of the other.

Use Applet ML19.A
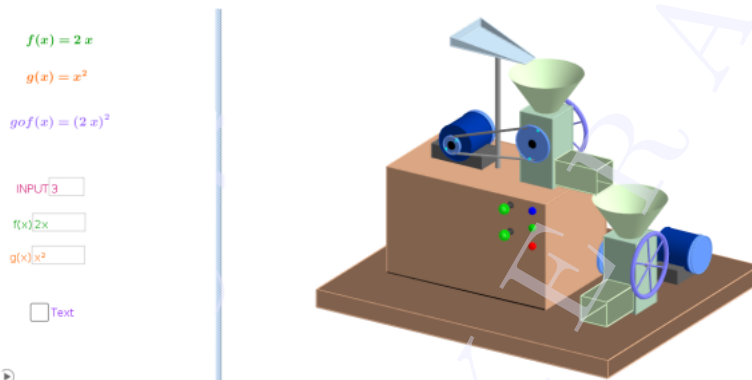
About the applet

In 3D graphics we can see two machines combined together. Output of the first machine is the input of the second. There are three buttons on the machine, Blue,Green and Red,in order to refresh, start or stop the machine. Two indicator lights are provided which turns red if the corresponding input is not suitable for the machines. From the Graphics view we can edit the functions as well as the input number

Procedure

- Set $f(x) = 2x$ and $g(x) = x^2$

  📖 Find the values of $g \circ f(2)$ and $g \circ f(3)$ and $g \circ f(-3)$

- Set $f(x) = 2x$ and $g(x) = \sqrt{x}$

  📖 Find the values of $g \circ f(2)$ and $g \circ f(3)$

  📖 What happens to the indicators if we give -3 as the input? why?

  📖 What can you say about the domain of the function g in order to define $g \circ f$?

- Set $f(x) = \sqrt{x}$ and $g(x) = x^2$

  📖 Find the values of $g \circ f(2)$ and $g \circ f(1)$ and $g \circ f(0)$

  📖 What happens to the indicators if we give -2 as the input? why?

  📖 Find the domain in which $g \circ f(x) = x$

  📖 Find the domain in which $f \circ g(x) = x$.

Lab 20

Inverse Trigonometric Functions

Aim

- To make trigonometric functions invertible by restricting the domain and co - domain with the help of their graphs

- To draw the graphs of inverse trigonometric functions

- To find the value of inverse trigonometric functions from graphs of corresponding trigonometric function and from unit circle

Concepts

- Trigonometric functions

- Inverse trigonometric functions

Discussion

We discuss how to restrict domain and co - domain of a trigonometric function so as to make it invertible

Activity 20.1 Making Invertible

As we have seen in Activity 18.3 we can make a function one to one and onto by restricting its domain and co-domain. In this activity we discuss trigonometric functions and make them bijections.

Procedure

- Mark the distance on x axis in terms of $\frac{\pi}{2}$. (Right click on Graphics View $\rightarrow$ Graphics $\rightarrow$ x axis $\rightarrow$ Distance $\rightarrow$ Select $\dfrac{\pi}{2}$ from the drop down menu)

- Draw the graph of $f(x) = \sin x$.

- By observing the graph we can see that if we redefine f as $f : [\frac{-\pi}{2}, \frac{\pi}{2}] \rightarrow [-1, 1]$ it becomes a bijection.

- Draw the graph of the above function (use the command `If(-pi/2<=x<=pi/2,f)`.)
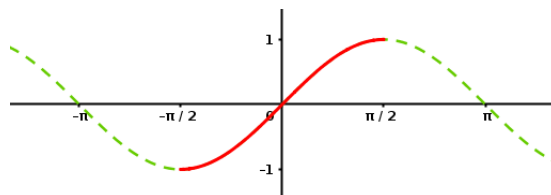
- 📖   By selecting suitable domain and co-domain, make all trigonometric functions invertible. Draw the corresponding graphs and complete the following table (There are infinitely many possible domains which makes a trigonometric function bijective. You may select any one among them)

Function	Domain	Range
$\sin x$	$\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$	$[-1, 1]$
$\cos x$		
$\tan x$		
$\operatorname{cosec} x$		
$\sec x$		
$\cot x$		

- Hide all graphs and save the GeoGebra file as Activity 20.1

- 📖   Draw the graphs of the following functions, select suitable domain and co-domain to make them invertible. Draw their graphs in the restricted domain. (You can make necessary change in the marking on the x axis)

  i) $\sin 2x$ ii) $2\cos 3x$ iii) $\sec 2x$ iv) $\sin\left(\frac{x}{2}\right)$

Activity 20.2 Graph of Inverse Trigonometric Functions

We know that the graph of f^{-1} is obtained by reflecting the graph of f on the line $y = x$. We use this idea to draw the graph of inverse trigonometric functions.

- Open the GeoGebra file Activity 20.1
- Mark the distance on x axis as
  . . . $-1, 0, 1, 2, 3, . . .$ and that on y axis in terms of $\frac{\pi}{2}$
- Draw the line $y = x$
- Open the graphs of trigonometric functions (bijective) one by one and reflect them on the line $y = x$ (using `Reflect about Line` tool.)
- save the file as Activity 20.2
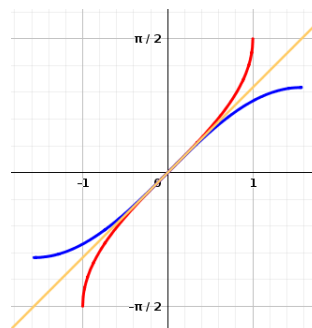
- 📖   Write the domain and range of the of the inverse trigonometric functions.

Function	Domain	Range
$\sin^{-1} x$		
$\cos^{-1} x$		
$\tan^{-1} x$		
$\operatorname{cosec}^{-1} x$		
$\sec^{-1} x$		
$\cot^{-1} x$		

We can draw the graphs of inverse trigonometric functions using input commands

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Function	Command
$\sin^{-1} x$	`sin^-1(x)` or `asin(x)`
$\cos^{-1} x$	`cos^-1(x)` or `acos(x)`
$\tan^{-1} x$	`tan^-1(x)` or `atan(x)`
$\csc^{-1} x$	`cosec^-1(x)` or `asin(1/x)`
$\sec^{-1} x$	`sec^-1(x)` or `acos(1/x)`
$\cot^{-1} x$	`cot^-1(x)` or `pi/2-atan(x)`

Activity 20.3 Value of Inverse Trigonometric Functions from Graph

In Activity 19.1 we discussed the method of finding $f^{-1}(a)$, from the graph of the function f, for given values of a. We use this method to find the values of inverse trigonometric functions from the graphs of the corresponding trigonometric functions.

Use the applet ML20.3

About the applet

In this applet graph of a function f is given. Using an input box we can edit the function. Using another input box we can change the value of the slider **a**. The point A on the y-axis represents $(0, a)$. x co ordinate of the point C represents

$f^{-1}(a)$
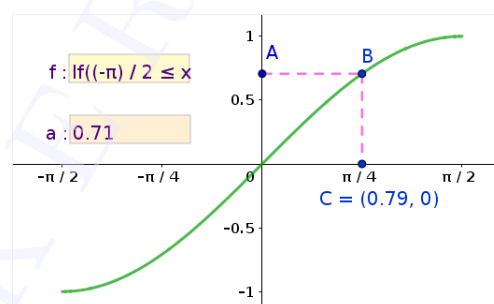
📖 Find the values of $\sin^{-1}(0.5)$ and $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

📖 Find the values of the following. If possible write in terms of π otherwise write in decimal form (may be an approximate value)

$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$	$\cos^{-1}(0.5)$	$\tan^{-1}(5)$	$\csc^{-1}(-2)$	$\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$	$\cot^{-1}(-4)$

Activity 20.4 Value of Inverse Trigonometric Functions from Unit Circle

We have defined $\sin x$ and $\cos x$ with the help of a unit circle centered at the origin. Here we use the reverse of that process to find the values of $\sin^{-1} x$ and $\cos^{-1} x$ for given values of x.

Use applet ML20.4

About the applet

- For a given value of a we can find the values of $\sin^{-1} a$, $\cos^{-1} a$ or $\tan^{-1} a$. We can select the required one using the check boxes.

- We can change the value of a using the slider

- If we select the check box ' sine' we get a semi circle of radius 1 on the positive side of the x axis and a point B, with y co-ordinate a, moves along y axis according to a.

Procedure

- Suppose we are asked to find $\sin 0.85$ (refer the figure given below), we rotate the point P from $A(1, 0)$ by an arc of length 0.85 (or by an angle of 0.85 radian) in anticlockwise direction. The y coordinate of P gives $\sin 0.85$ that is 0.75

  If we are asked to find $\sin^{-1} 0.75$, then the question is reversed, that is, what should be the

angle (or arc length) of rotation of P in order to get its y coordinate as 0.75 ?. So we plot the point $B(0, 0.75)$, the corresponding point on the circle and find the arc length (or angle). So from the figure $\sin^{-1} 0.75 = 0.85$ (It may be an approximate value). If the rotation of P is in the clockwise direction we take the value as negative.
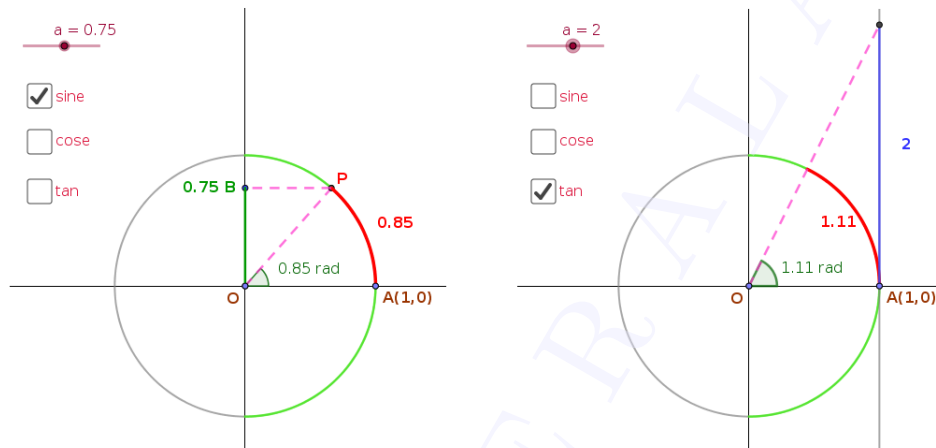
- In a similar manner we can find $\cos^{-1} a$ for give values of a

- 📖 The figure given above shows that $\tan^{-1} 2 = 1.11$. Discuss the reason

- 📖 Discuss a method to find $\tan^{-1} a$, using the applet, for given values of a

- 📖 Using the applet find the following values

$\sin^{-1}(0.4)$	$\sin^{-1}(-0.6)$	$\cos^{-1}(0.8)$	$\cos^{-1}(-0.5)$	$\tan^{-1}(5)$	$\tan^{-1}(-3)$

Lab 21

Properties of Inverse Trigonometric Functions

Aim

- To derive identities involving inverse trigonometric functions graphically

Concepts

- Graphs of trigonometric functions

- Trigonometric identities

Discussion

We have studied some trigonometric identities in class 11. Here we derive some identities involving inverse trigonometric functions with the help of graphs of corresponding trigonometric functions

Activity 21.1

Use the applet ML21.1

About the applet

- Graphs of the functions $f : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$ defined by $f(x) = \sin x$ and
  $g : [-\frac{\pi}{2}, \frac{\pi}{2}] - \{0\} \to \mathbb{R} - (-1, 1)$ defined by $g(x) =$ cosec x are given.

- A point is given on the x axis. Its position is given as y. We can change the position using the slider

- $x = g(y)$ and $x_1 = f(y)$ are given on the y axis

- We can redefine f and g using input boxes
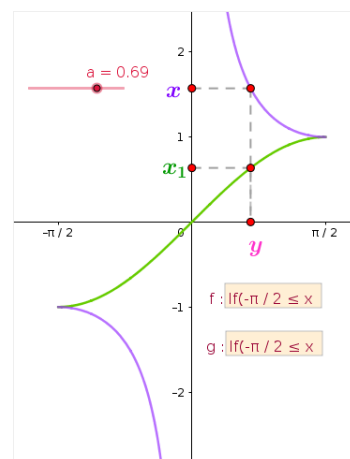
Procedure

- 📖 Find the relation between x and x_1

- 📖 We have $\operatorname{cosec}^{-1} x = y$ and $\sin^{-1} x_1 = y$. Using this deduce that

$\sin^{-1} \frac{1}{x} = \operatorname{cosec}^{-1} x, \ x \geq 1$ or $x \leq -1$

- 📖 Does this result or any other result similar to this holds for $x \in (-1, 1)$?

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- 📖 Deduce the results

  $\cos^{-1}\frac{1}{x} = \sec^{-1}x,\ x \geq 1$ or $x \leq -1$ and

  $\tan^{-1}\frac{1}{x} = \cot^{-1}x,\ x > 0$. What happens if $x \leq 0$

Activity 21.2

Use the applet ML21.2

About the applet

- Graphs of the functions $f : [0, \pi] \to [-1, 1]$,defined by $f(x) = \cos x$ is given.

- We can change the function using the input box

- x and $-x$ are given on the y axis. We can change their position using the slider **a**

- $y = \cos^{-1} x$ and $y_1 = \cos^{-1}(-x)$ are given on the x axis.

Procedure

- Initially set the value of the slider at 0 and gradually increase it

- 📖 Observe the movements of y and y_1 and find the relation between them

- 📖 Find the relation between $\cos^{-1} x$ and $\cos^{-1}(-x)$
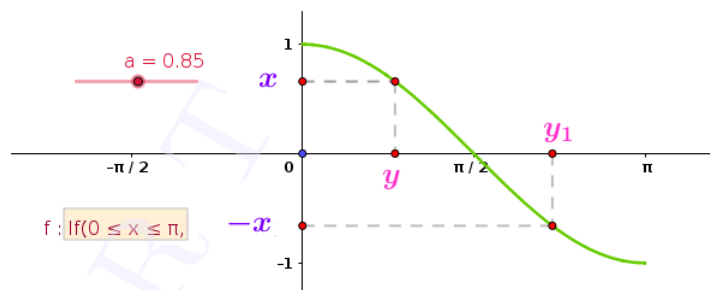
- 📖 Similar results holds for other inverse trigonometric functions also. Derive them.

Activity 21.3

Use the applet ML21.3

About the applet

- Graphs of the functions $f(x) : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$ defined by $f(x) = \sin x$ and
  $g(x) : [0, \pi] \to [-1, 1]$ defined by $g(x) = \cos x$ are given

- $y_1 = \sin^{-1} x$ and $y_2 = \cos^{-1} x$. We can change the position of x using the slider **a**

Procedure :

- Initially set the value of the slider at 0 and gradually change it.

- 📖   Observe the movements of y_1 and y_2 and find the relation between them

- 📖   Find the relation between $\sin^{-1} x$ and $\cos^{-1} x$

- 📖   Similar result holds among $\tan^{-1} x$, $\cot^{-1} x$, $\operatorname{cosec}^{-1} x$ and $\sec^{-1} x$. Derive the results
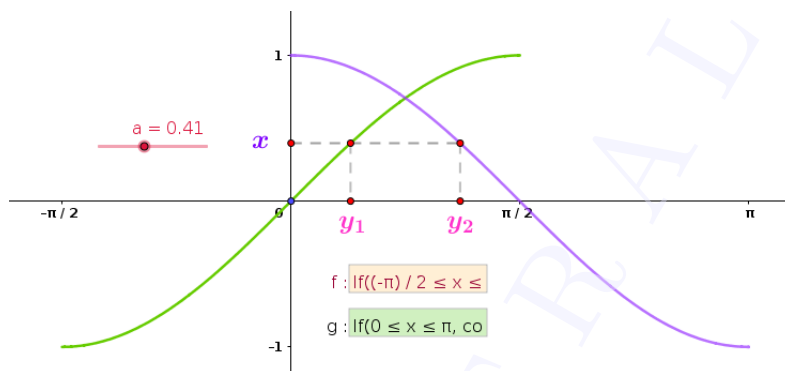
Activity 21.4

Draw the graphs of $\sin(\sin^{-1} x)$ and $\sin^{-1}(\sin x)$. Are they same ? Find their domain.

We know that $\sin(\sin^{-1} x) = x$ in $[-1, 1]$ and $\sin^{-1}(\sin x) = x$ in $[-\frac{\pi}{2}, \frac{\pi}{2}]$. i.e. both are identity functions, but on different domains. First function is defined only on $[-1, 1]$ but the second one is defined for all real numbers. In this activity we discuss the definition of $\sin^{-1}(\sin x)$ on $\mathbb{R}$

Use the applet ML31.4

About the applet

- Graphs of the functions $f : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$ defined by $f(x) = \sin x$ and $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = \sin x$ are given

- We can change the positions of x and x_1, given on the x axis using the slider **a**. The position of y will change accordingly.

Procedure :

- In the applet $x_1 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and x is any real number

- We have $\sin x_1 = \sin x = y$
  So $\sin^{-1} \sin x_1 = \sin^{-1} \sin x = \sin^{-1} y = x_1$ .... (1)

- If $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ then $x = x_1$. So $\sin^{-1} \sin x = x_1 = x$ (from (1))

  That is

  $\sin^{-1} \sin x = x$,   if $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

- 📖   Let $x \in [\frac{\pi}{2}, \frac{3\pi}{2}]$. Adjust the value of **a** so that $x = \pi$. Then gradually increase or decease the value of **a**. Observe the movement of x and x_1. Find the relation between them.

- We can see that $x_1 = \pi - x$. So $\sin^{-1} \sin x = x_1 = \pi - x$

  That is

  $\sin^{-1} \sin x = \pi - x$,   if $x \in [\frac{\pi}{2}, \frac{3\pi}{2}]$
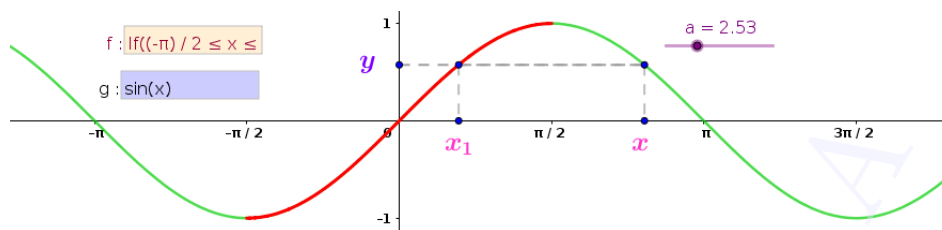
- 📖 Write the definition of $\sin^{-1} \sin x$ if $x \in [\frac{3\pi}{2}, \frac{5\pi}{2}]$

- 📖 Write the definition of $\sin^{-1} \sin x$ if $x \in \mathbb{R}$

- 📖 Write the definitions of $\cos^{-1} \cos x$ and $\tan^{-1} \tan x$ if $x \in \mathbb{R}$

Lab 22

Inverse trigonometric functions - Visualisation of problems

Aim

- To visuslise identities involving inverse trigonometric functions geometrically.

- To derive properties of inverse trigonometric functions geometrically.

Concepts

- Properties of inverse trigonometric functions.

Discussion

We have derived some properties of inverse trigonometric functions in some intervals. In this lab we visualise this properties with the help of graphs of inverse trigonometric functions and extend the properties to other intervals

Activity 22.1

Procedure

Mark the distance on x axis annd y axis in terms of $\dfrac{\pi}{4}$.

📖 Draw the graphs of the following functions. Write the simplest form of the functions in the given intervals.

- $\tan^{-1}\left(\dfrac{\cos 2x}{1 - \sin 2x}\right), \quad -\dfrac{3\pi}{4} < x < \dfrac{\pi}{4}$

- $\tan^{-1}\left(\dfrac{1 + \tan x}{1 - \tan x}\right), \quad -\dfrac{3\pi}{4} < x < \dfrac{\pi}{4}$

- $\tan^{-1}\left(\dfrac{\cos x - \sin x}{\cos x + \sin x}\right), \quad -\dfrac{\pi}{4} < x < \dfrac{3\pi}{4}$

- $\tan^{-1}\left(\sqrt{\dfrac{1 - \cos x}{1 + \cos x}}\right), \quad 0 < x < \pi$

📖 Prove the results using algebraic methods.

Activity 22.2

procedure

- Draw the graph of $\sin^{-1}(\sin(x))$.

    📖   Identify the interval in which $\sin^{-1}(\sin(x)) = x$.

    📖   Find $\sin^{-1}(\sin x)$ in other intervals.
    (Hint : Shifting of the graphs of $f(x) = x$ or $f(x) = -x$)
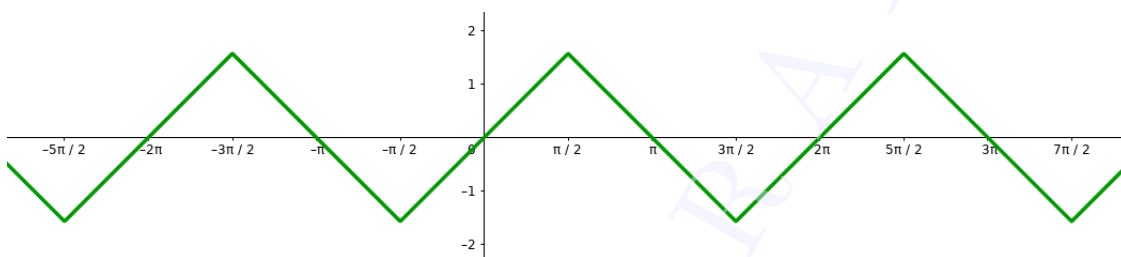
📖   Find the values of the following by algebraic methods and verify your answer from the graph of $\sin^{-1}(\sin x)$.(Mark the distance on x axis and y axis in terms of $\frac{\pi}{12}$)

i) $\sin^{-1}(\sin \frac{7\pi}{6})$ ii) $\sin^{-1}(\sin \frac{5\pi}{3})$ iii) $\sin^{-1}(\sin \frac{5\pi}{4})$

Activity 22.3

procedure

- Mark the distance on x axis annd y axis in terms of $\dfrac{\pi}{4}$.

    📖   Draw the graphs of $\cos^{-1}(\cos x)$ and $\tan^{-1}(\tan x)$. Identify the intervals in which they coincides with the identity function. Find their definition in other intervals.

    📖   Find the values of the following by algebraic methods and verify your answer from graphs.

    i) $\cos^{-1}(\cos \frac{4\pi}{3})$ ii) $\cos^{-1}(\cos \frac{5\pi}{2})$ iii) $\tan^{-1}(\tan \frac{7\pi}{6})$ iv) $\tan^{-1}(\tan \frac{5\pi}{3})$

Activity 22.4

procedure

    📖   Using algebraic methods show that $\tan^{-1}\left(\dfrac{1}{\sqrt{x^2 - 1}}\right) = \operatorname{cosec}^{-1}x, x > 1$

- Draw the graphs of $\tan^{-1}\left(\dfrac{1}{\sqrt{x^2 - 1}}\right)$ and $\operatorname{cosec}^{-1}x$ and verify your result.

    📖   Observe the graphs and find the simplest form of $\tan^{-1}\left(\dfrac{1}{\sqrt{x^2 - 1}}\right)$ for $x < -1$

Additional Activities

Activity 22.A

Procedure

- Draw the graph of $f(x) = \sin^{-1}(2x\sqrt{1-x^2})$ and
  $g(x) = 2\sin^{-1}x$ and verify that

  $f(x) = g(x)$ when $-\dfrac{1}{\sqrt{2}} \leq x \leq \dfrac{1}{\sqrt{2}}$

  📖   Write $\sin^{-1}(2x\sqrt{1-x^2})$ in terms of $2\sin^{-1}x$ in
  $\frac{1}{\sqrt{2}} \leq x \leq 1$ (Hint: Draw the graph of $-g$, give a vertical
  shift to $-g$ so that it coincides with f in $\frac{1}{\sqrt{2}} \leq x \leq 1$. It will
  be more convinient to mark the distance on y axis as $\frac{\pi}{2}$)

  📖   Write $\sin^{-1}(2x\sqrt{1-x^2})$ in terms of $\cos^{-1}x$ in
  $\frac{1}{\sqrt{2}} \leq x \leq 1$

  📖   Write $\sin^{-1}(2x\sqrt{1-x^2})$ in terms of $2\sin^{-1}x$ in
  $-1 \leq x \leq -\frac{1}{\sqrt{2}}$
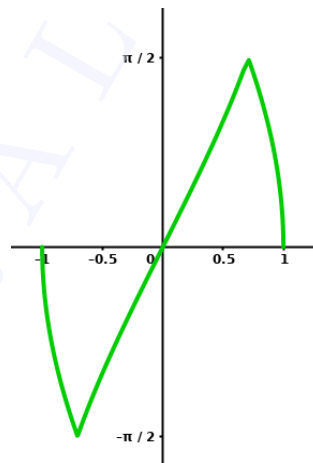
Activity 22.B

procedure

- Verify the result $2\tan^{-1}x = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$, $|x| \leq 1$ by drawing the graphs of

  $f(x) = 2\tan^{-1}x$ and $g(x) = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

  📖   Write $2\tan^{-1}x$ in terms of $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$ for $x > 1$ and $x < -1$ (Hint: Draw the
  graph of $-g$,give a vertical shift to $-g$ so that it coincides with f for $x > 1$ or $x < -1$.It will
  be more convinient to mark the distance in terms of $\dfrac{\pi}{2}$)

  📖   With the help of graphs, write $2\tan^{-1}x$ in terms of $\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

  📖   With the help of graphs, write $2\tan^{-1}x$ in terms of $\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$

  📖   With the help of graphs, write $\sin^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ in terms of $2\tan^{-1}x$

  📖   With the help of graphs, write $\cos^{-1}\left(\dfrac{2x}{1+x^2}\right)$ in terms of $2\tan^{-1}x$

Activity 22.C

- Draw the graphs of the following functions. Identify the intervals in which they coincides
  with the identity function. Find their definition in other intervals.

  i) $\operatorname{cosec}^{-1}(\operatorname{cosec} x)$ ii) $\sec^{-1}(\sec x)$ iii) $\cot^{-1}(\cot x)$

Lab 23

Matrices and Determinants

Aim

- To explore the properties of matrix multiplication

- To explore the properties of determinants

- To solve system of linear equations

Concepts

- Matrix operations

- Determinants and properties of determinants

- Solution of a system of linear equations

Discussion

We discuss different methods of creation of matrices, matrix operations and determinants using GeoGebra. With the help of the software it is easy to verify properties of matrix multiplication and properties of determinants. We also discuss solution of a system of linear equations

Activity 23.1 Formation of Matrix Using GeoGebra

We can create a matrix by the following ways

Input Method

Suppose we want to create the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 4 & 6 \end{bmatrix}$$

Give the input command

`A={{1,2,3},{3,2,1},{2,4,6}}`

Spreadsheet Method

- Open Spreadsheet (View → Spreadsheet)

- Type the numbers as shown in the figure given below

- Select the Matrix tool. If required we can edit the name

- By Clicking 'Create'. We can see the matrix in the Algebra view.
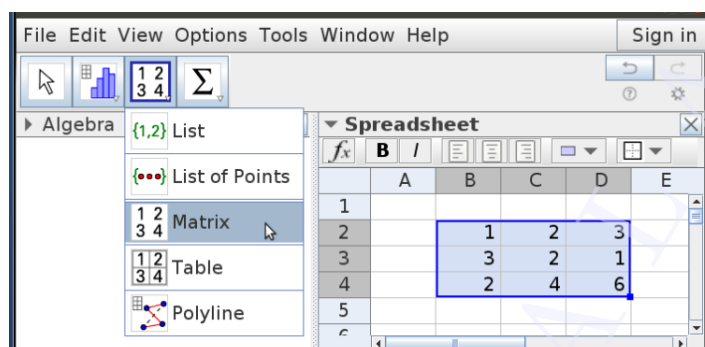
- Another way of creating the matrix is that Select the numbers entered → Right click → Create → Matrix

Matrix operations using GeoGebra commands

- If A is any matrix and k is a scalar, then the input command `k*A` gives the matrix kA

- If A and B are two matrices of the same order, then The input command $A + B$ gives their sum and $A - B$ gives their difference.

- The input command `Transpose(A)` gives the transpose of the matrix A

- `A*B` gives the product of the matrices A and B and `A^2` gives the square of the matrix A

- Inverse of a matrix A is obtained by the input command Invert(A)

Activity 23.2 Multiplication of Matrices

Procedure
- Create matrices A of order 3×2 and B of order 2×3 and find their product[We can give the command $AB = A * B$]

- 📖   Double the entries of the first row of A. What happens to the entries of AB ?

- Replace the second raw of A by the sum of its first two rows (we may use the notation $R_2 \to R_1 + R_2$)

- 📖   Triple the entries of the second column of B. Observe the change in AB

> To observe the step by step changes in the elements of AB,Use Record to Spreadsheet tool.For this right click on AB and select Record to Spreadsheet

- 📖   Complete the following table

A	B	AB
$R_1 \to 2R_1$	No change	$R_1 \to 2R_1$
$R_3 \to -R_3$	No change	
$R_2 \to R_1 + 2R_2$	No change	
No change	$C_2 \to 3C_2$	
No change	$C_1 \to C_1 + C_2$	

- Create matrices A of order 3×2 and B of order 2×3.

- 📖   Verify that $(AB)' = B'A'$

- 📖   Do this for different matrices A and B by editing their entries

Activity 23.3 Formation of Determinants

If A is a square matrix then the input command `Determinant(A)` gives $|A|$.

Procedure:

📖   Create the 2×2 square matrix $A = [a_{ij}]$ where $a_{ij} = i + 2j$ and find $|A|$

📖   Create the 3×3 square matrix $A = [a_{ij}]$ where $a_{ij} = 2i - j$ and find $|A|$

- Create an integer slider k.

- Create square matrices A, B, C, D of order 1, 2, 3 and 4 respectively.

- Create the matrices kA, kB, kC and kD.

- Find the determinants of the six matrices created.

- Note :

  - We can find the determinants of $kA, kB, ...$ even without creating the matrices.
  - If we give the inputs like `detA=Determinant(A)`, `detkA=Determinant(kA)` it will be easy to distinguish them from the algebra view.

📖   What is the relation between $|A|$ and $|kA|$. Verify your answer for different values of k and different matrices A.

📖   Similarly find the relation between $|B|$ and $|kB|$.

📖   Complete the following table.

Order of matrix	k	$|M|$	$|kM|$	Relation between $|M|$ and $|kM|$
1	1	$|A| = ...$	$|kA| = ...$	$|kA| = ...|A|$
2				
3				
4				

- Note : M is any one of the matrices A, B, C and D.

- 📖   If M is a square matrix of order n and k is a real number, what is the relation between $|M|$ and $|kM|$?. Why ?

Activity 23.4 Properties of Determinants

Procedure:

Interchanging rows and columns

- Create a 3×3 matrix A and its transpose.

- Find their determinants.

  📖   What is the relation between $|A|$ and $|A'|$?

- Try the same with different matrices.

Interchanging two rows

- Create a 3×3 matrix A as follows.

  – Enter nine numbers in the form of a 3×3 matrix.

  – Create a list from the first row as follows

    Select the first row of the three numbers $\rightarrow$ Right click $\rightarrow$ Create $\rightarrow$ List

  – Rename the list as R_1 (if the default name is different)

  – Similarly creates the lists R_2 and R_3 from the second and third rows respectively.

  – Now create the matrix A using the input command $A = \{R_1, R_2, R_3\}$

- Create matrix B, obtained from A by interchanging its first two rows. Use the input command. $B = \{R_2, R_1, R_3\}$

- Find $|A|$ and $|B|$.

  📖   Find the relation between $|A|$ and $|B|$. Confirm your answer for different matrices A.

- Create an input box for B

- Redefine B, by interchanging the second and third rows of A. ($\{R_1, R_3, R_2\}$)

  📖   Find the relation between $|A|$ and $|B|$.

  📖   If any two rows are interchanged then what happens to a determinant ?

- Save the applet as Activity 23.2A

Cyclic Change

- In the above applet, redefine B as $\{R_2, R_3, R_1\}$

  📖   Find the relation between $|A|$ and $|B|$. Confirm your answer for different matrices A.

  📖   Redefine B as $\{R_3, R_1, R_2\}$ and find the relation between $|A|$ and $|B|$.

  📖   What happens to a determinant if its rows are cyclically changed ?

- Save the applet as Activity 23.2B

Scalar Multiplication

- In the above applet, create a slider k

- Redefine B as $\{k * R_1, R_2, R_3\}$, so that it is obtained from A by multiplying its first row by k.

  📖   Find the relation between $|A|$ and $|B|$. Confirm your answer for different values of k and different matrices A.

  📖   Redefine B as $\{R_1, R_2, k * R_3\}$ and find the relation between $|A|$ and $|B|$.

  📖   If a row of a determined is multiplied by a constant then what happens to its value ?

- Redefine B as $\{k * R_1, k * R_2, R_3\}$, so that it is obtained from A by multiplying its first and second rows by k.

  📖   Find the relation between $|A|$ and $|B|$.

- Redefine B as `k*A`. That is $B = kA$.

  📖   Find the relation between $|A|$ and $|B|$. Confirm your answer for different values of k and different matrices A.

  📖   If A is a matrix of order n and k is a constant, what is the relation between $|A|$ and $|kA|$. Why ?

- Save the applet as Activity 23.2C

Identical Rows

- Use the applet Activity 23.2A

- Edit B as $\{R_1, R_1, R_3\}$

  📖   Find the value of $|B|$. Do this for different matrices B

  📖   Change B as $\{R_1, R_2, R_1\}$ and find $|B|$

  📖   What is the value of a determinant if its any two rows are identical ?

Linear combination

- Use the applet Activity 23.2C

- Edit B as $\{R_1, R_2 + k * R_1, R_3\}$. That is B is obtained from A by applying the operation $R_2 \to R_2 + kR_1$.

  📖   What is the relation between A and B ?. Confirm your answer for different values of k and different matrices A.

- Edit B as $\{R_1 + kR_3, R_2, R_3\}$ and repeat the activity.

  📖   In general, what do you infer from this activity ?

  📖   What happens to the determinant if any of the rows is a linear combination of others?. For example $R_3 = R_1 + kR_2$

Additional Activities

Activity 23.A Apply Matrix

Linear algebra is the crux of computer graphics.In a computer screen each point can be represented with two numbers (co-ordinates). These two numbers can be represented as a column matrix say $\begin{bmatrix} a \\ b \end{bmatrix}$,this 2×1 matrix when multiplied to a 2×2 matrix will result in a column matrix of order 2×1,which can be represented again by another point in the screen.In other words we can say that the first point was transformed to the second point by the matrix multiplication.

In this activity we explore such transformations and nature of the matrix for various transformations like reflection,rotation etc.Instead of looking at transformation of a single point,we can also apply transformations to a set of points ,which representing a curve or figure

Basics

- A point $P(a, b)$ is represented by the matrix $\begin{bmatrix} a \\ b \end{bmatrix}$

- If $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$, the command *ApplyMatrix(A,P)*, transforms the P to the point represented by the matrix AP .ie $(pa + qb, ra + sb)$

- The command *ApplyMatrix(A,k)*,where k is any object(a point or a curve,or a polygon or a picture...)

  Procedure

- Create the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$

  - Plot the point $P(-1, 2)$
  - Find manually the transformed position of the point P when we give the command *ApplyMatrix(A,P)*
  - Verify your answer by inputting the command

- Create a slider k and edit above matrix A as $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ .Apply the command on the following objects and observe the change for different values of k

  - The point $(3, 2)$
  - A regular pentagon (Using regular polygon tool we can construct a pentagon)
  - Unit circle centered at origin
  - Circle of radius 2 centered at $(1, 2)$
  - A picture inserted(Use the image tool to insert a picture)

- Edit above matrix A to $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$. Apply the command on above objects and observe the change in different value of k

- Edit above matrix A to $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ .Apply the command on above objects and observe the changes

- Find the matrix A to get the following results on applying the command

  - The object reflects about the x-axis
  - The object reflect about the origin
  - Get the projection of the object on x-axis
  - Verify your answers using the command

- Rotation

  If P is the point $(r\cos x, r\sin x)$ and A is the matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ .Then the command

  ApplyMatrix(A,P) gives the point represented by $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} r\cos x \\ r\sin x \end{bmatrix}$

  That is the point $(r\cos(x+\theta), r\sin(x+\theta))$,the point obtained by rotating P by an angle θ

- Create a slider θ and the matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

  Apply the command on the following objects , animate the slider and observe the movement of the objects

  - The point (3,2)
  - Circle of radius 1 unit centered at (4,0)
  - A picture inserted

- Creating a slider **a**, and creating the matrix $\frac{a}{5}A$ can you give the point P , a spiral rotation?

Lab 24

Applications of
Matrices and Determinants

Aim

- To explore some applications of matrices and determinants

Concepts

- Matrix operations

- Determinants and properties of determinants

- The method of finding area of triangle if the coordinates of its vertices are given

- Solution of a system of linear equations

Discussion

We discuss some applications of matrices and determinants such as area of triangle and solution of a system of linear equations in two variables and three variables. We also use the help of geometry in the above discussion

Activity 24.1 Area of triangle

We know that the area of a triangle whose vertices are $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) is given by

$$\Delta = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

1. Find the area of the triangle whose vertices are $(2, 2), (8, 4), (4, 6)$

   Procedure

   - Let us construct the corresponding matrix using the input command
     `M={{2,2,1},{8,4,1},{4,6,1}}`
   - Find the value of the determinant using the input command `Det(M)`
   - Half of the above value gives the area.
   - Draw the triangle using Polygon Tool , find the area and verify your answer.

2. $A(2, 3)$ and $B(8, 2)$ are vertices of triangle ABC having area 10 square units. Find coordinates of the third vertex C.

Procedure

- We can take the point C as (x, y)

  Since the area is 10

  $\frac{1}{2}\begin{vmatrix} 2 & 3 & 1 \\ 8 & 2 & 1 \\ x & y & 1 \end{vmatrix} = 10$ We can expand the determinant as follows

- Give the input command `M={{2,3,1},{8,2,1},{x,y,1}}`

- Open the CAS window ($View \longrightarrow CAS$)and give the input command `Expand(Determinant(M))`. This gives a linear expression in x and y.

- Equate above expression to 20 (why ?)to get a linear equation in x and y.

- Draw the above line

- Plot the points A and B and plot a point C on the line.

- Draw the triangle ABC and find its area.

  📖 Change the position of C. what happens to the area ? Why it happens so ? Give a geometrical explanation.

  📖 Is any position of C, other than points on the above line, gives the same area ? If so why did we miss it ?

3. Vertices of two triangles are given as $(2, 1), (7, 4), (5, 7)$ and $(4, 1), (14, 4), (10, 7)$. Look at a glance at the coordinates and predict the relation between their areas, with out actually calculating them.(Hint : You can use a property of determinants). Draw the triangles and verify your answer.

4. $A(-2, 1), B(5, 3)$ and $C(1, 6)$ are the vertices of triangle ABC. Write the coordinates of a triangle, whose area is three times the area of triangle ABC. Draw the triangles and verify your answer.

5. Draw a triangle and find its area. Using move tool click inside the triangle and drag it. Observe that its area is not changing (it is quite natural). Give an explanation for this using properties of determinants.

6. Draw a triangle such that either the x coordinates of the vertices are even or the y coordinates are even (or both). Find the area and observe that it is an integer. Explain the reason using properties of determinants.

7. Draw a triangle such that its x and y coordinates are odd.Find the area and observe that it is an integer. Explain the reason using properties of determinants.

Activity 24.2 Solution of a System of Linear Equations in two variables

If a system of linear equations is represented by $AX = B$, where A is a non singular matrix, then the solution is given by $X = A^{-1}B$.

1. Solve the following system of equations using matrices. Draw the corresponding lines and verify your answer.

$$2x + 3y = 12$$
$$3x - 2y = 5$$

- Create the matrices $A = \begin{bmatrix} 2 & 3 \\ 3 & -2 \end{bmatrix}$ (input command : `A={{2,3},{3,-2}}`

  and $B = \begin{bmatrix} 12 \\ 5 \end{bmatrix}$ (input command : `B={{12},{5}}`)

- Find A^{-1} (input command `M=Invert(A)`)

  📖 Find $A^{-1}B$ (input command: `M*B`) and hence find the solution of the given system of equations.

- Draw the lines $2x + 3y = 12$ and $3x - 2y = 5$. Find their point of intersectin and verify above answer.

2. Solve the following system of equations using matrices.

$$6x + 4y = 15$$
$$9x + 6y = 5$$

- Create the matrices $A = \begin{bmatrix} 6 & 4 \\ 9 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 15 \\ 5 \end{bmatrix}$
- Find A^{-1} , What happens ? Why it so ?
- Find $|A|$ (Input command `Determinant(A)`)
- We know that if A is singular, the consistancy of the systen deppends on the nature of the matrix $adj(A)B$.

  📖 Find $adj(A)B$ and comment about the number of solutions. Draw the lines and verify your answer.

3. Examine the consistency of the system of equations.

$$6x + 12y = 15$$
$$4x + 8y = 10$$

- Create the matrices A and B
- Find $|A|$. If it is zero, find $adj(A)B$. Do you get a conclusion about the consistency of the system of equations ?

  📖 Draw the lines and comment about the consistency of the system of equations.

4. Two lines in a plane are either intersecting or coinciding parallel (with out coinciding).

  Consider the lines

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

- They are intersecting if they are not parallel, That is if $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$

- They are coinciding if $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

- They are parallel, but not coinciding if $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} \neq \dfrac{c_1}{c_2}$

  📖 Using this, explain the relation between the number of solutions of a system of equations in two variables the value of $|A|$ and the nature of $adj(A)B$

Activity 24.3 Solution of a System of Linear Equations in three variables

Find the solution of the following system of equations.

$$3x - 2y + 3z = 8$$
$$2x + y - z = 1$$
$$4x - 3y + 2z = 4$$

Procedure:

- Create the matrices $A = \begin{bmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 8 \\ 1 \\ 4 \end{bmatrix}$

- Find A^{-1}

  📖   Find $A^{-1}B$ and hence find the solution of the given system of equations.

  📖   Find the solution of the following system of equations.

  i)

$$x - y + z = 4$$
$$2x + y - 3z = 0$$
$$x + y + z = 2$$

  ii)

$$x - y + 2z = 7$$
$$3x + 4y - 5z + 5 = 0$$
$$2x - y + 3z = 12$$

Additional Activities

Activity 24.A The Geometry of Solution of System of Equations **

Consider solving the system of equations

$$x - 2y + 3z = 1$$
$$x + 2y - z = 13$$
$$3x + 2y - 5z = 3$$

First this system of equations is to be converted to the matrix form represented by $AX = B$ as

$$\begin{bmatrix} 1 & -2 & 3 \\ 1 & 2 & -1 \\ 3 & 2 & -5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 13 \\ 3 \end{bmatrix}$$

Procedure

- Create the matrix A

- Create the matrix B

- Find the inverse of the matrix A using the invert command.The matrix is m_1

- Find the product of m_1 and b in the input bar

  📖   Note down the solution of the system of equations

- Open the 3D Graphics view

- In the input bar type the first equation $x - 2y + 3z = 1$.A plane eq1 is formed in the Graphics view

- In the input bar type the second equation $x + 2y - z = 13$.A plane eq2 is formed in the Graphics view

- In the input bar type the third equation $3x + 2y - 5z = 3$.A plane eq3 is formed in the Graphics view

- Using the intersect command intersect the two equations eq1 and eq2 .A line f is formed when the two planes represented by the equations intersect

- Using the intersect command intersect the line f and the equation eq1.A point is formed

  📖   What do you observe regarding the co-ordinates of this point and the matrix m_2 obtained earlier.Can you give a mathematical reasoning for this?
  Now let us consider the system of equations represented by

$$x + 2y - 2z = 1$$
$$2x + 2y - z = 6$$
$$3x + 4y - 3z = 5$$

- Repeat the above procedure for these set of equations

  📖   What do you observe regarding the solution of this set of equations

  📖   Can you find the geometrical reason why the solution is peculiar here?
  Now let us consider the system of equations represented by

$$x - y - 5z = -2$$
$$2x + y + 4z = 2$$
$$2x + 4y - 2z = 8$$

- Repeat the above procedure for these set of equations

  📖   What do you observe regarding the solution of this set of equations

  📖   Can you find the geometrical reason why the solution is peculiar here?
  (**To be attempted after completing 3D)

Lab 25

Continuity

Aim

- To explore the concept of continuity of a function at a point

Concepts

- Limit of a function at a point

- Continuity at a point

Discussion

We discuss the continuity of a function at a point graphically. We also discuss the continuity of some special functions

Activity 25.1 Continuity from graph

Procedure

Use the applet ML25.1
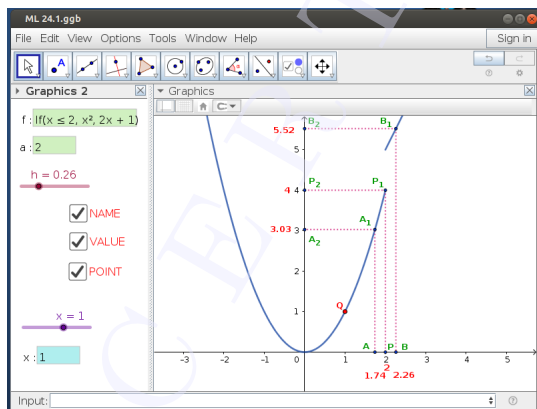
About the applet

- You can see the graph of a function $f(x)$, 3 points A, B, P on the x axis, corresponding points A_1, B_1, P_1 on the graph and A_2, B_2, P_2 on the y axis

- NAME Check Box: By clicking on it you can show/hide the names of the points

- VALUE Check Box: By clicking on it you can show/hide the x coordinates of the points A, B and P and the y coordinates of the points A_2, B_2 and P_2

- POINT Check Box : By clicking on it we get a point Q on the graph. A slider and an input box named **x** appear on Graphics 2. Using them we can change the position of the point Q.

- Slider **h**: Using this we can bring the points A and B towards P

- Input box **a**: To change the position of P

- Input box **f**: To change the function

Continuity at a point

- Change the function as

$$f(x) = \begin{cases} x^2 & \text{if } x \le 2 \\ 2x + 1 & \text{if } x > 2 \end{cases}$$

> We can find the limit of the function f at the point a using the command $Limit(f, a)$

  📖 Discuss its limit at $x = 2$

  📖 Show the point Q. Using slider **x** move the point and observe its movement. What happens to it when x crosses 2 ?

  📖 Discuss the continuity of f

- Change the function as

$$f(x) = \begin{cases} x^2 & \text{if } x \ne 2 \\ 5 & \text{if } x = 2 \end{cases}$$

  📖 Discuss the limit of f at $x = 2$

  📖 Observe the movement of Q as x crosses 2.

  📖 Discuss the continuity of f

- Discuss the continuity of the following functions

  i) $f(x) = \begin{cases} x^2 + 2 & \text{if } x \le 0 \\ -x^2 - 2 & \text{if } x > 0 \end{cases}$ at $x = 0$

  ii) $f(x) = \begin{cases} x^3 - 4 & \text{if } x < 2 \\ x^2 - 2 & \text{if } x > 2 \\ 5 & \text{if } x = 2 \end{cases}$ at $x = 2$

- Save the function as Activity25.1

Activity 25.2 The function $\sin \frac{1}{x}$

Procedure

- Change the function $f(x)$ as $f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$

- Observe the movement of Q as x approaches 0

  📖 Discuss the existence of the limit and continuity of the function at $x = 0$

- Change the function $f(x)$ as

  $$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$$

- Observe the movement of Q as x approaches 0, what is the difference from the previous case, Why it is so ?

  📖 Discuss the existence of the limit and continuity of the function at $x = 0$

  📖 Similarly Discuss the existence of the limit and continuity of the function

  f defined as $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$ at $x = 0$
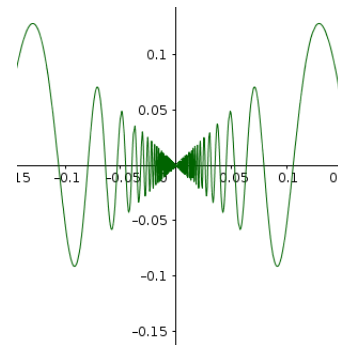
Activity 25.3 Make it Continuous

Consider the function
$$f(x) = \begin{cases} kx + 1 & \text{if } x \leq 5 \\ 3x - 5 & \text{if } x > 5 \end{cases}$$
Using GeoGebra we can find the value of k for which f is continuous.

Procedure

- Create a slider k with Min $= -5$ and Max $= 10$

- Construct the graph of the function
  $$f(x) = \begin{cases} kx + 1 & \text{if } x \leq 5 \\ 3x - 5 & \text{if } x > 5 \end{cases}$$
  Use the command `If(x ≤ 5, k x + 1, 3x - 5)`

- By changing the value of k we can join the two pieces of the curve.

  📖   Find the value of k so that the function becomes continuous.

- Note: To check the accuracy of k, we can zoom the curve at point where two pieces of the curve meet.

- Create an input box for the function.

  📖   Find the value of k, so that the following functions are continuous.

i) $f(x) = \begin{cases} kx^2 & \text{if } x \leq 2 \\ 3 & \text{if } x > 2 \end{cases}$ ii) $f(x) = \begin{cases} \dfrac{k \cos x}{\pi - 2x} & \text{if } x \neq \dfrac{\pi}{2} \\ 3 & \text{if } x = \dfrac{\pi}{2} \end{cases}$

iii) $f(x) = \begin{cases} k(x^2 - 2x) & \text{if } x \leq 0 \\ 4x + 1 & \text{if } x > 0 \end{cases}$

Verify your answer by algebraic method.

Lab 26

Differentiability

Aim

- To visualise the derivative of e^x

- To explore the geometrical meaning of derivative of implicit functions

- To explore the geometrical meaning of derivative of functions in parametric form

- To explore Rolle's theorem and Mean Value theorem.

Concepts

- Derivative of a function

- Implicit function

- Functions in parametric form

- Geometrical meaning of derivative of a function at a point

- Rolle's theorem and Mean Value theorem

Discussion

We discuss the derivative of e^x geometrically. We also discuss the geometrical meaning of derivative of implicit functions and functions in parametric form. Geometrical meaning of Rolle's theorem and Mean Value theorem are also discussed

Activity 26.1 Derivative of e^x

We discuss a geometrical visualisation of the derivative of e^x.

Procedure

- Create a slider **a** with Min = 1, Max = 100 and Increment = 0.01.

- Draw the graph of $f(x) = a^x$.

- Draw the graph of $f'(x)$.

- Change the value of **a** and observe the graphs.
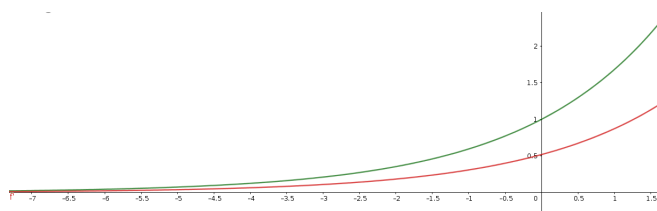
📖   Find an approximate value of **a** for which the graphs coincide. Zoom the graph and decrease the increment of the slider to get better approximation.

Activity 26.2 Derivatives of Implicit Functions

1. We discuss the geometrical meaning of implicit functions and their derivatives.

Consider the equation $x^2 + y^2 = 25$.

By usual method we can see that $\dfrac{dy}{dx} = -\dfrac{x}{y}$

What is the value of $\dfrac{dy}{dx}$ when $x = 4$?

From the equation $x^2 + y^2 = 25$ we can see that

$$x = 4 \Longrightarrow y = \pm 3$$

$$\Longrightarrow \frac{dy}{dx} = \pm\frac{4}{3}$$

Why we get two values for $\dfrac{dy}{dx}$?

Let us have a geometrical approach to this matter.

Procedure

- Draw the curve $x^2 + y^2 = 25$. We know that it represents a circle.

- Create a slider with **a** with Min = -10, Max = 10 and increment 0.01

- Draw the line $x = a$ and plot its poins of intersection A and B with the circle.

- Draw tangents to the circle at A and B and find their slopes.

- Compare the slopes with the values of $\dfrac{dy}{dx}$ that we got. (The input command x(A)/y(A) gives the value of $\dfrac{x}{y}$ at the point A.)

- Change the value of the slider **a**, compare the slopes of the tangents with values of derivative.

- What happens to the tangents and the values of $\dfrac{dy}{dx}$ when $a = \pm 5$, $a > 5$ or $a < 5$.

  Here the equation $x^2 + y^2 = 25$ does't represent a function but it implies two functions.
  $x^2 + y^2 = 25 \Longrightarrow y = \sqrt{25 - x^2}$ and $y = -\sqrt{25 - x^2}$

- Draw the graph of these functions. What do they represent ?

- Save this appltet as Activity26.2

Note :

Using GeoGebra we can find the derivative of an implicit function in the following ways.

- Input k=x^2+y^2-25. We can see it in the Algebra View as a multi variable function $k(x, y) = x^2 + y^2 - 25$.

- The input command ImplicitDerivative(k) gives $\dfrac{dy}{dx}$.

- We can also give the input command directly as ImplicitDerivative(x^2+y^2-25)

2. In the above applet change the equation of the curve as $x^2 + xy + y^2 = 12$. Find $\dfrac{dy}{dx}$ and find its values for different values of x. Change the slider **a** accordingly and compare $\dfrac{dy}{dx}$ with the slope of the tangents. What happens to the values of $\dfrac{dy}{dx}$ and to the tangents for $a = 0$ and $a = \pm 4$. Try to answer algebraically and verify using the applet.

3. Change the equation of the curve as $x^2 + xy - y^2 = 5$ and do the above activity. What happens when $a = \pm 2$

Activity 26.3 Parametric curves

We have seen that there are two tangents to the circle $x^2 + y^2 = 25$, corresponding to $x = 3$. We can distinguish them as 'Tangent at the point $(3, 4)$' or 'Tangent at the point $(3, -4)$'
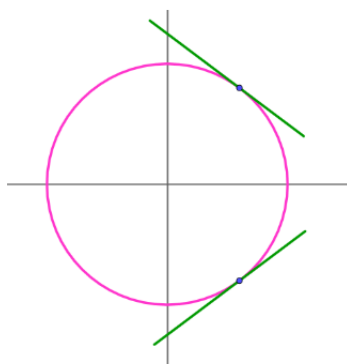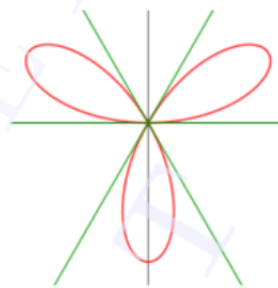
fig 1   fig 2

In fig 2 there are three tangents to the curve at the origin. We can't distinguish them by simply saying 'Tangent at $(0,0)$'. In some cases even the representation of such curves as algebraic expressions in x and y is not easy. We use another method, called parametric equation of the curve.

Consider the equations

$$x = 5\cos(t)$$
$$y = 5\sin(t)$$

What does it represent ?. Let us try to plot the curve.

Procedure

- Create a slider **a** with Min = 0, Max = pi and Increment = 0.001.

- Plot the point $A(5\sin(t), 5\cos(t))$ and give its trace.

- Animate the slider and observe the path of the point A. What does it represent ?

- We can eliminate t between the equations as follows.

$$x^2 + y^2 = 25\sin^2(t) + 25\cos^2(t)$$
$$\implies \quad x^2 + y^2 = 25$$

- Whe know that this represents a circle.

- We can represent any point on this circle in terms of t

  📖 Find the values of t for which the point reaches at $(3, 4)$ and $(3, -4)$.

The equations

$$x = 5\cos(t)$$
$$y = 5\sin(t)$$

are called the paremetric equations of the circle $x^2 + y^2 = 25$. 't' is called the parameter. Similarly the parametric equations of the curve in fig 2 are

$$x = 4\sin(3t)\cos(t)$$
$$y = 4\sin(3t)\sin(t)$$

Procedure

- Edit the poin A as $(4\sin(3t)\cos(t), 4\sin(3t)\sin(t))$ and plot the curve by animating the slider.

- We can draw the curve directly using the input command `a=Curve(4sin(3t)cos(t),4sin(3t)sin(t),t,0,pi)` (Instead of t we can use any variable). Draw the curve using above command.

- We can see the point A as it moves along the curve, crosses the origin three times resulting three tangents at $(0,0)$. How can we draw these tangents?(tangent tool or command gives only one tangent)

- To draw the tangents we need their slopes, which is given by $\dfrac{dy}{dx}$. We can find it using the input command `ParametricDerivative(a)` which gives a new parametric curve as shown below.

$$\text{b : } \begin{array}{l} x = 4\ \sin(3\ t)\ \cos(t) \\[4pt] y = \dfrac{\cos(t)\cdot 4\ \sin(3\ t) + 4\cdot 3\ \cos(3\ t)\ \sin(t)}{\sin(t)\ (-1)\cdot 4\ \sin(3\ t) + 4\cdot 3\ \cos(3\ t)\ \cos(t)} \end{array} \Bigg\}\ 0 \le t \le 3.14$$

In this y gives the derivative $\dfrac{dy}{dx}$ (Verify this by usual methods)

- Now we have to find $\dfrac{dy}{dx}$ for the values of t for which the point A reaches (0,0).

  This is obtained by solving the following equations

$$x = \sin 3t\ \cos t = 0$$
$$y = \sin 3t\ \sin t = 0$$

(Squaring and adding $\implies \sin^2 3t = 0 \implies t = 0, \frac{\pi}{3}, \frac{2\pi}{3}$)

- To get the value of $\dfrac{dy}{dx}$ at $t = 0$ give the input command `b(0)`. This gives a new point B in the Algebra View, whose y coordinate gives the value of $\dfrac{dy}{dx}$ at $t = 0$. Similarly find its values at $t = \frac{\pi}{3}, \frac{2\pi}{3}$
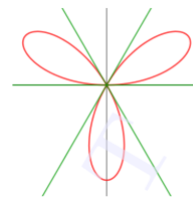
  📖 Find the equations of the tangents and draw them.

👉 The curve is said to have multiple tangents at $(0,0)$

📖 Draw the following curves .Find the points at which the curve has a vertical tangent or a horizontal tangent or multiple tangents. Find their equations and draw the tangents.

- $x = 2\cos t, y = \sin 2t$ (Input `Curve(2cos(t),sin(2t),t,0,2pi)`)

- $x = t^2, y = t^3 - 3t$ (Input `Curve(t^2,t^3-3t,t,-pi,pi)`)

Activity 26.4 Rolle's Theorem

If a function satisfies the conditions of Rolle's theorem in $[a, b]$, then there exists $c \in (a, b)$ such that $f'(c) = 0$. Geometrically there exist a point on the curve, between $(a, f(a))$ and $(b, f(b))$ at which the tangent is parallel to the x axis

Check whether Rolle's theorem is applicable for the function $x^2 - 4x + 6$ $x \in [0, 4]$. If it is applicable find the value of c and verify Rolle's theorem.

Procedure

- Draw the graph of the function $f(x) = x^2 - 4x + 6$ in [0,4]. (Input command `f=If(0<=x<=4,x^2-4x+6)`)

- Check whether $f(0) = f(4)$

- Check the continuity and differentiability by observing the graph.

- Create a slider **c** and plot the point $C(c, f(c)$

- Draw the tangent to the curve at C and find its slope.

📖 Find the value of c for which the slope of the tangent is 0, i.e. the tangent is parallel to the x axis.

📖 Check whether Rolle's theorem is applicable for the following functions. If it is applicable find the value of c and verify Rolle's theorem.

1) $f(x) = x^2 + 2$ $\qquad$ $x \in [-2, 2]$ $\qquad$ 2) $f(x) = x^2 - 1$ $\qquad$ $x \in [1, 2]]$

3) $f(x) = x^2 + 2x - 8$ $\qquad$ $x \in [-4, 2]$ $\qquad$ 4) $f(x) = |\sin x|$ $\qquad$ $x \in [0, 2\pi]$

Activity 26.5 Mean Value Theorem

If a function satisfies the conditions of Mean value theorem in $[a, b]$, then there exists a point on the curve, between $(a, f(a))$ and $(b, f(b))$ at which the tangent is parallel to the chord joining above points

Verify Mean Value theorem for the function
$f(x) = x^2 - 4x - 3$ in $[1, 4]$

Procedure

- Draw the graph of $f(x)$

- Plot the points $A(1, f(1))$ and $B(4, f(4))$

- Draw the segment AB and find its slope.

- Create a slider **c** and plot the point $C(c, f(c))$.

- Draw the tangent to the curve at C and find its slope.

📖 Find the value of c so that the slope of the tangent line is equal to that of the secant line. Hence verify Mean Value theorem.

📖 Verify Mean Value theorem for the following functions.

1) $f(x) = x(x - 2)$ $\qquad$ $x \in [1, 3]$ $\qquad$ 2) $f(x) = x^2 - 4x - 3$ $\qquad$ $x \in [1, 4]]$

3) $f(x) = x - 2\sin x$ $\qquad$ $x \in [-\pi, \pi]$ $\qquad$ 4) $f(x) = (x - 3)^2$ $\qquad$ $x \in [3, 4]$
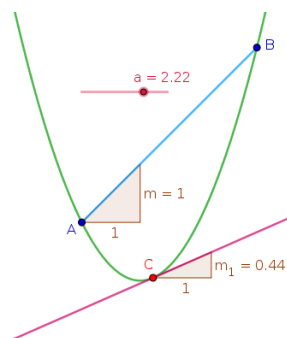
Additional Activities

Activity 26.A Derivative of e^x

Considering $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \cdots$ we discuss its derivative.

Procedure

- Create an integer slider **m** with Min $= 0$ and Max $= 100$

- Create the graph of the function $f(x) = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \cdots + \dfrac{x^m}{m!}$

  using the input command `f=Sum(x^n/n!,n,0,m)`

- Draw the graph of $f'(x)$. (Input : `f'`).
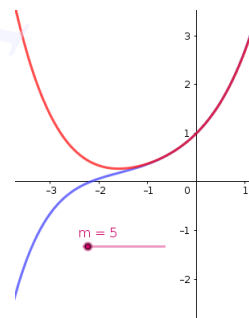
📖   As **m** increases to ∞ what happens to the function $f(x)$?

📖   What happens to the graphs of $f(x)$ and $f'(x)$ as **m** in creases?

📖   From this activity what do you infer about the derivative of e^x ?

Activity 26.B Parametric Curves

📖   Create three sliders **a**, **b** with Min $= 0$ and slider **c** with Min $= 0$ and Max $= 50$. Using this sliders draw the following curves. Observe the curves and find the values of **c** for which $\dfrac{dy}{dx} = 0$ or not defined. Verify your answer by finding $\dfrac{dy}{dx}$ in usual method

1) $x = at^2, y = 2at$ (Input: `Curve(a*t^2,2a*t,t,-c,c)`

2) $x = a\cos\theta, y = b\sin\theta$ (Input: `Curve(a*cos(t),b*sin(t),t,0,c)`

Change the values of **a** and **b** and observe the change in the shape of the curve

3) $x = \cos\theta - \cos 2\theta, y = \sin\theta - \sin 2\theta$

(Input: `Curve(cos(t)-cos(2t),sin(t)-sin(2t),t,0,c)`)

4) $x = a(\theta + \sin\theta), y = a(1 - \cos\theta)$

(Input: `Curve(a(t+sin(t)),a(1-cos(t)),t,0,c)`)

5) $x = a(\cos\theta + \theta\sin\theta), y = a(\sin\theta - \theta\cos\theta)$

(Input: `Curve(a(cos(t)+t*sin(t)),a(sin(t)-t*cos(t)),t,0,c)`
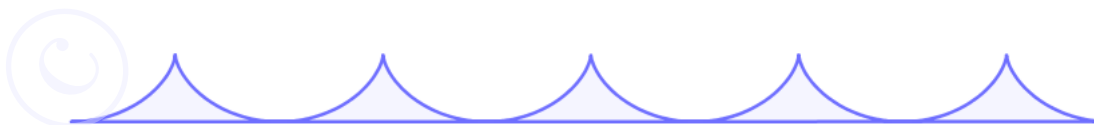
Create an integer slider **n**. Using the given input command, draw the following curves

1) Curve(n*cos(t)^3,n*sin(t)^3,t,0,2pi)

2) Curve(sin(t),cos(nt),t,0,2pi)

3) Curve(cos(t)-cos(nt),sin(t)-sin(nt),t,0,2pi)

4) Curve((e^cos(t)-3cos(4t)+sin(t/12)^5)sin(t), (e^cos(t)-3cos(4t)+sin(t/12)^5)cos(t),t,0,2pi)
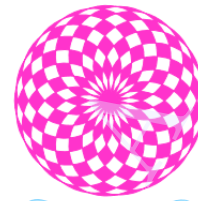
Lab 27

Increasing and Decreasing Functions

Aim

- To explore increasing and decreasing functions geometrically

Concepts

- Increasing and decreasing function

- Derivative of a function and slope of tangent at a point on a curve

Discussion

We discuss the definition of increasing and decreasing functions graphically. We find the intervals in which a function is increasing or decreasing by mere observation of the graph of function, and by observing the the graph of its derivative function

Activity 27.1 Increasing and Decreasing Functions 1

We discuss the definition of increasing / decreasing functions

Procedure

- Draw the graph of the function $f(x) = \dfrac{1}{3}x^3 - \dfrac{1}{2}x^2 - 2x$.

- Create a slider **a** with Min $= -5$ Max $= 5$ and Increment 0.01

- Plot the points $A(a, f(a))$ and $B(0, f(a))$. Join AB.

- Gradually change the value of the slider from -5 to 5 and observe the movement of the point B.

  📖 Describe the motion of B as **a** increases from $-\infty$ to ∞.

  📖 Find the intervals in which the function f is increasing / decreasing

  📖 Observe the graphs of the following functions and find the intervals in which they are increasing / decreasing. (You can do it by mere observation of the graphs. In the case of trigonometric functions change the markings on x axis in terms of π).
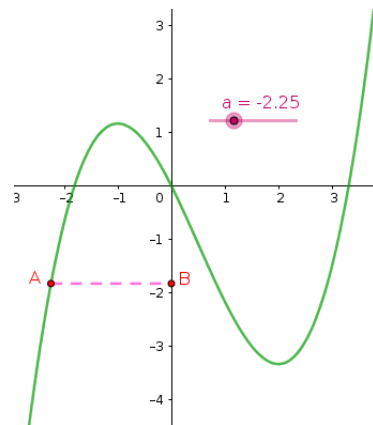
1. $x^2 - 4x + 6$

2. $\sin x + \cos x,\ x \in [0, 2\pi]$

3. $x^3 - 3x^2 + 4x$

Activity 27.2 Increasing and Decreasing Functions 2

We derive the property of the derivative of a function in the intervals at which the function is increasing or decreasing.

Procedure

- Draw the graph of the function discussed in the above activity

- Create a slider **a** and plot the point $A(a, f(a))$

- Draw the tangent to the curve at A and find its slope.

  📖 Find the intervals in which the slope of the tangent is positive / negative.

- Do the activity with the function $f(x) = \dfrac{1}{x}$.

  📖 Find the relation between the sign of $f'(x)$ and the intervals in which the function f is increasing / decreasing
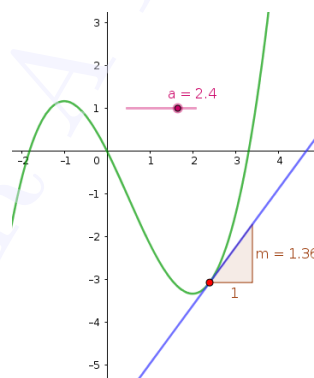
Activity 27.3 Increasing and Decreasing Functions 3

Sometimes it is not possible, by mere observation of the graph, to find the exact value at which the function turns from increasing to decreasing or vice versa. In such cases we can use the derivatives of the functions.

We use the concept that if a function f is differentiable at some point where it turns from increasing to decreasing or vice versa, then $f'(x) = 0$ at that point. We also use the property that we obtained from above activity

Procedure

- Draw the graph of the function $f(x) = 8x^3 - 9x^2 - 15x$

- Draw the graph of $f'(x)$

- Plot the points at which the graph of $f'(x)$ meets the x axis. Show the coordinates of these points.

  📖 Find the intervals in which the function is increasing / decreasing

  📖 Find the intervals in which the following functions are increasing / strictly decreasing.

1. $8x^3 - 9x^2 - 15x$

2. $\sin 3x$

3. $(x+1)^3(x-3)^3$

4. $x^3 - 3x^2 + 3x - 100$

Lab 28

Tangents and Normals 1

Aim

• To explore the geometry of tangents and normals to a curve

Concepts

• Tangent to a curve at a point

• Normal to a curve at a point

Discussion

By definition tangent to a curve at a point is the line passing through that point and having slope equal to the derivative at that point,if it exist. But the tangent may exist even if the derivative does not exist.

We discuss the following peculiarities of the tangents using different examples

• Touches the curve at a single point

• Touches the curve at more than one points

• Cuts the curve at a point different from the point of contact

• Cuts the curve at the point of contact

Activity 28.1 Tangent 1

• Draw the graph of $f(x) = x^2$ and plot a point A on it.

• Draw the tangent to the curve at A (use `Tangents` tool). Move the point and observe the tangent.

   📖 Write the peculiarity of the tangent (as discribed in the discussion).

• Change the function as $f(x) = x^3$. Move the point and observe the tangent.

   📖 Write the peculiarity of the tangent.

   📖 What happens to the tangent at $A(0,0)$?
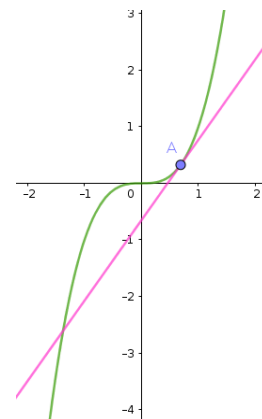
Activity 28.2 Tangent 2

Procedure

- Draw the graph of $\sin x$ and plot the point $A\left(\dfrac{\pi}{2}, 1\right)$

- Draw the tangent to the curve at A.
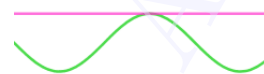
📖 Observe the tangent and write its peculiarity.

📖 Is there any other point on the curve at which the tangent has the same peculiarity ?

📖 Give another example for a curve and a point on it at which the tangent has the same peculiarity.

📖 Is there any point on the curve $y = \sin x$ at which the tangent crosses the curve at that point ?

Activity 28.3 Vertical Tangent

Procedure

- Draw the curve $x = y^2$

- Create a slider **a** and plot the point $A(a^2, a)$.

- Draw the tangent to the curve at A and find its slope.

- Change the value of **a** and observe the slope of the tangent.
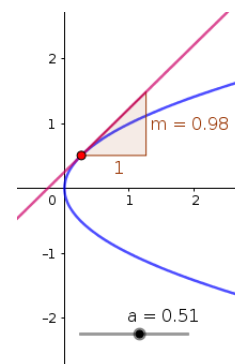
📖 What happens to the slope of the tangent at $(0, 0)$.

📖 What is the derivative $\left(\dfrac{dy}{dx}\right)$ at $(0, 0)$

📖 Draw the following curves, observe them and find the points at which they have vertical tangents.

   i) $x^2 + y^2 = 16$ ii) $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$

Activity 28.4 Normal

Normal to a curve at a point is defined as the line passing through that point and perpendicular to the tangent at that point (if the tangent exists). We discuss the method of construction of normal.

Procedure

- Draw the graph of the function $f(x) = x^2$ and plot a point A on it.

- Create a slider **a** and plot the point $A(a, f(a))$.

- Draw the tangent to the curve at A

- Using `Perpendicular Line` tool draw the normal to the curve at A.

  📖   Edit the function as $f(x) = x^3$. Change the position of A and observe the tangent and normal. Is it possible to have a normal to this curve with positive slope? What happens to the tangent and normal at $(0, 0)$?
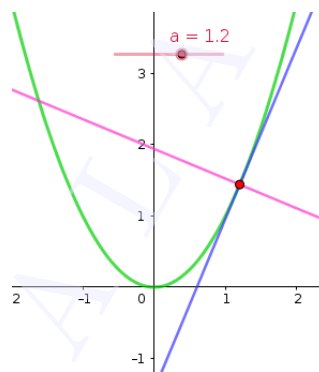
Activity 28.5 Tangent and Normal to a Circle

Procedure

- Draw a circle and plot a point on it.

- Draw the tangent and normal to the circle at that point.

- Change the position of the point and observe the normal.

  📖   What is the peculiarity of the normal ?

  📖   Give an independent definition (not in terms of tangent) of normal to a circle at a point on it.

  📖   Define tangent to a circle at a point on it, without using 'derivative'

Lab 29

Tangents and Normals 2

Aim

- To explore geometrically the concept of tangents and normals at a given point to a curve

- To explore the concept of tangents and normals when the curve is given in parametric form

- To visualise problems involving tangents and normals

Concepts

- Equation of tangent

- Equation of normal

Discussion

We discuss the tangent and normal of a curve at a given point with the help of GeoGebra, We also discuss the tangents and normals to a curve when it is expressed in parametric form.

Activity 29.1 Tangents and Normals to a Curve at a Point

Using `Tangents` tool we can draw the tangent to a curve at a given point.Then using `Perpendicular Line` tool we can draw the normal.

Procedure

Answer the following questions with the help of GeoGebra.

1. Find the slope of the tangent to the curve $y = \dfrac{x-1}{x-2}, x \neq 2$ at $x = 10$

   <u>Procedure</u>

   - Draw the graph of $f(x) = \dfrac{x-1}{x-2}$.
   - Plot the point $A(10, f(10))$.
   - Draw the tangent to the curve at A and find its slope

2. Find the slope of the tangent to the curve $y = x^3 - 3x + 2$ at the point whose x coordinate is 3.

3. Find the equation of the tangent to the curve $\dfrac{x-7}{(x-2)(x-3)}$ at the point where it cuts the

   x axis. Find the equation by usual methods and compare the answers.

4. Find the equations of the tangent and normal to the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 2$, at $(1, 1)$.
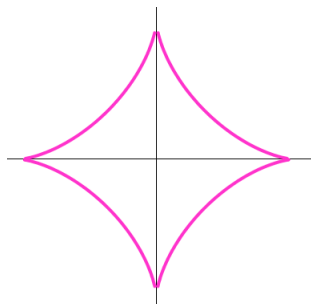
Activity 29.2 Parametric forms

Find the slope of tangent and normal to the following curves using GeoGebra.

1. $x = a \sin^3 t, \quad y = a \cos^3 t \qquad$ at $\quad t = \dfrac{3\pi}{4}$.

    What about the existence of the tangent at $t = \dfrac{\pi}{2}$?

    Procedure

    - Create a slider **a**
    - Draw the curve using the the input `c=Curve(a*sin^3(t),a*cos^3(t),t,0,2pi)`.
    - Plot the point where $t = \dfrac{3\pi}{4}$ using the input command `c(3pi/4)`.

        📖 Draw tangent and normal to the curve at the point and find their slopes.

    - Change the value of the slider and observe the movement of the tangent and the normal.

2. $x = at^2, \quad y = 2at \qquad$ at $\quad t = 1$

3. $x = \cos t, \quad y = \sin t \qquad$ at $\quad t = \dfrac{\pi}{4}$

4. $x = 1 - a \sin t, \quad y = a \cos^2 t \qquad$ at $\quad t = \pi$. What about the existence of tangent at $t = \dfrac{\pi}{4}$?

Activity 29.3 Problems involving Tangents and Normals

Answer the following questions with the help of GeoGebra.

1. Find points at which the tangent to the curve $y = x^3 - 3x^2 - 9x + 7$ is parallel to the x-axis.

   Procedure

   - The derivative of the function is zero where the tangent is parallel to the x axis. Here we use this idea.

   - Draw the graph of the function $f(x) = x^3 - 3x^2 - 9x + 7$

   - Draw the graph of $f'(x)$

   - Plot the points (say A, B $\cdots$) at which $f'(x)$ meets the x axis.

   - The input command $(x(A), f(x(A))$ gives a point on the curve $y = f(x)$ at which tangent is parallel to the x axis.

   - Similarly find the other points.

   - Save the applet as ML29.3 so that we can use it for doing similar problems by editing the function.

2. Find the equations of all lines having slope 0 which are tangent to the curve $y = \dfrac{1}{x^2 - 2x + 3}$

3. Find the point at which the tangent to the curve $y = \sqrt{4x - 3} - 1$ has its slope $\dfrac{2}{3}$.

4. Find the point on the curve $y = x^3 - 11x + 5$ at which the tangent is $y = x - 11$

5. Find the equation of all lines having slope 2 which are tangents to the curve $y = \dfrac{1}{x - 3}, x \neq 3$

6. For the curve $y = 4x^3 - 2x^5$, find all the points at which the tangent passes through the origin.
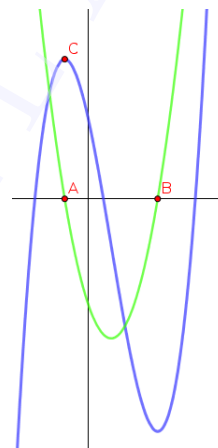
Additional Activities

Activity 29.A Tangent line and Secant line

Draw the tangent to the curve $y = (x-2)^2$ at which the tangent is parallel to the chord joining the points $(1, 1)$ and $(4, 4)$.

Procedure

- Here we have to find the points on the curve where the derivative is equal to the slope of the secant line.

- Draw the graph of $f(x) = (x-2)^2$ and plot the points $A(1, f(1)$ and $B(4, f(4))$ on it.

- Draw the secant line AB and find its slope m.

- Create a slider **c** with increment 0.01

- Draw the line $y = mx + c$
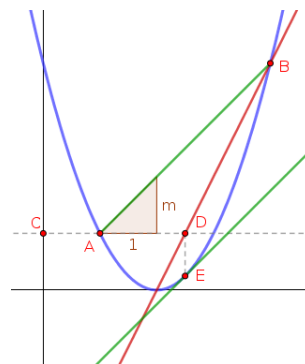
- Find the points of intersection of the line with the curve.

- Adjust the value of **c** so that the points of intersection coinside each other and the secant line becomes tangent to the curve.

- Write the point of tangency and the equation of the tangent

Lab 30

Maxima and Minima

Aim

- To explore the concept of local maxima and local minima

- To find points of local maxima, local minima, absolute maximum and absolute minimum geometrically

Concepts

- Local maxima and local minima

- Absolute maximum and absolute minimum

Discussion

We discuss the method of finding points of local maxima, local minima, local maximum value, local minimum value, absolute maximum and absolute minimum with the help of graph of a function and graph of its derivative.

Activity 30.1 Local Maxima and Local Minima

If a function has a local maximum or a local minimum at a point and if the function is differentiable at that point, then the derivative at that point is zero. So it is enough to search for points of local maxima or local minima at the points where either the derivative equal to zero or the function is not differentiable.

📖 Find the points of local maxima and local minima of the function $f(x) = 8x^3 - 33x^2 + 36x - 8$ Also find local maximum and local minimum values.

<u>Procedure</u>

First Derivative test

- Draw the graph of $f(x)$ and $f'(x)$.

- Find the points A and B at which the graph of $f'(x)$ cuts the x axis. Show the coordinates of A and B.

- From the graph of $f(x)$ it is easy to identify whether the x coordinates of A and B are points of local maxima or local minima.

- The input commands `f(x(A))` and `f(x(B))` gives the corresponding values.

Second Derivative test

- Draw the graph of $f(x)$ and $f'(x)$ and $f''(x)$

- Find the points A and B at which the graph of $f'(x)$ cuts the x axis. Show the coordinates of A and B.

- Find the sign of $f''(x(A))$ and $f''(x(B))$. [This is possible by observing the graph of $f''(x)$ or by the input commands `f''(x(A))` and `f''(x(B))`].

📖 Find the points of local maxima and local minima of the function $|\sin(2x)|$
Also find the local maximum and local minimum values

Procedure

- Create an integer slider **n** and mark the distance on x axis as $\dfrac{\pi}{n}$. (Right click on the Graphics View → Graphics → x Axis → Distance → type `pi/n`).

- Draw the graphs of the functions $f(x) = |\sin(2x)|$ and $f'(x)$.

- Identify the points at which either $f(x)$ is not differentiable or $f'(x) = 0$.

- Find points of local maxima and local minima.
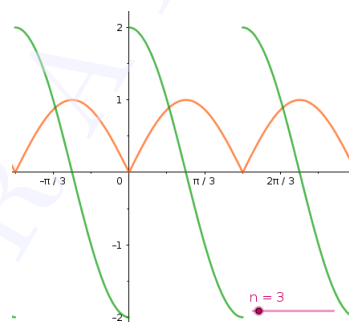
📖 Find the points of local maxima and local minima of the following functions
Also find the local maximum and local minimum values

1. $9x^2 + 12x + 2$

2. $x^3 + 1$

3. $\sin x - \cos x, 0 < x < \pi$

4. $\dfrac{1}{x^2 + 2}$

5. $\sin|x|$

Activity 30.2 Absolute Maximum and Absolute Minimum

📖 Find absolute maximum value and absolute minimum value of the following functions in the given intervals.

1. $f(x) = 2x^3 - 15x^2 + 36x + 1$ in [1,5]

   Procedure

   - Using the input command `Derivative(2x^3-15x^2+36x+1)` draw the graph of $f'(x)$.
   - Find the points A and B at which above graph meets x axis.
   - Find $f(1), f(5), f(x(A))$ and $f(x(B))$. Highest among them is the absolute maximum value and least is the absolute minimum value.

2. $3x^4 - 8x^3 + 12x^2 - 48x + 25$ in [0,3]

3. $\sin x + \cos x, \ x \in [0, \pi]$

Lab 31

Maxima and Minima - Visualisation of Problems

Aim

- To find points of local maxima, local minima, absolute maximum and absolute minimum geometrically

- To visualise some problems on maxima and minima

Concepts

- Local maxima and local minima

- Absolute maximum and absolute minimum

Discussion

We discuss the method of formulating a function from a given situation and finding maximum or minimum values with the help of graph of that function and graph of its derivative. We visualise some problems on maxima and minima by constructing GeoGebra applets.

Activity 31.1 Visualisation of Problems - 1

1. Using derivatives, find the maximum area of the rectangle inscribed in a circle of radius 2 units. Verify your answer using a GeoGebra construction.

   Procedure

   - Represent a side of the rectangle as x and express the area as function of x (say $f(x)$)

   - Draw the graphs of $f(x)$ and $f'(x)$

   - Plot the point(s) of intersection of the graph of $f'(x)$ with the x axis.

   - Identify the point at which $f(x)$ is maximum (say A).

     📖 $f(x(A))$ gives the maximum value of $f(x)$
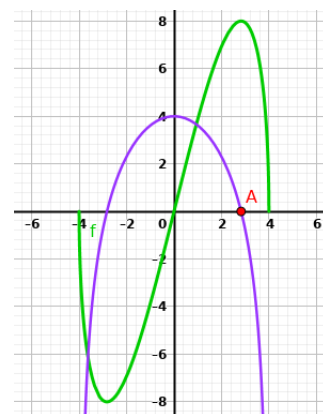
   We can Verify the result using the following GeoGebra construction

- Draw a circle of radius 2 centered at the origin

- Plot a point B on the circle.

- Plot the points $(-x(B), y(B)), (-x(B), -y(B))$ and $(x(B), -y(B))$

- Using *Polygon* tool, complete the rectangle joining the four points on the circle.

- Find the area of the rectangle.
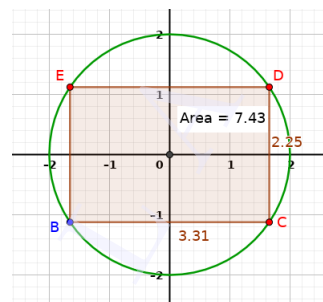
📖 Change the position of the point B and find the maximum area.

2. Find the minimum perimeter of rectangles having area 10.24. Verify your answer by constructing a GeoGebra applet.

Procedure

- Taking one of the sides of the rectangle as x and represent its perimeter as a function of x (say $f(x)$)

- Plot the graphs of $f(x)$ and $f'(x)$ and find the minimum perimeter as we have done in the last problem.

    Verification :

- Create a slider **a** with Min : 0, Max : 10 and Increment 0.1.

- Plot the points $(0,0), (a,0), (a, 10.24/a)$ and $(0, 10.24/a)$

- Complete the rectangle and find its perimeter. (Using Distance or Length tool click inside the rectangle)

📖 Change the value of the slider and find the minimum perimeter.

3. Find the maximum area of rectangles having perimeter 17.2. Verify your answer by constructing a GeoGebra applet.

Activity 31.2 Visualisation of Problems - 2

1. An Apache helicopter of enemy is flying along the curve given by $y = x^2 + 7$. A soldier, placed at $(3, 7)$, wants to shoot down the helicopter when it is nearest to him. Using derivatives find the nearest distance. Verify your answer by constructing a GeoGebra applet.

Procedure

- Take a point on the curve as $(x, x^2 + 7)$, write the distance of this point from $(3, 7)$ as a function of x (say $f(x)$)

- Draw the graphs of $f(x)$ and $f'(x)$

- Plot the point of intersection of the graph of $f'(x)$ with the x axis.

- Identify the point at which $f(x)$ is minimum (say A).

📖 $f(x(A))$ gives the minimum value of $f(x)$

We can Verify the result using the following GeoGebra construction

- Open a new GeoGebra window and draw the curve $y = x^2 + 7$

- Plot a point A on it

- Plot the point $B(3, 7)$

- Draw the line segment AB and find its length.

  📖 Change the position of the point A and find the minimum value of the length

2. Find the shortest distance of the curve $y = 2x^2 + 1$ from the point $(2, 1)$. Verify your answer by constructing a GeoGebra applet.

3. Find the shortest distance of the curve $y = x^3 + 2$ from the point $(4, 2)$. Verify your answer by constructing a GeoGebra applet.

Activity 31.3 Visualisation of Problems - 3

1. Find the maximum area of an isosceles triangle inscribed in the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$ with its vertex at one end of the major axis. Verify your answer by constructing a GeoGebra applet.

   Procedure

   - Taking $OP = x$, find PA and PQ in terms of x (use the equation of the ellipse)

   - Find the area of the triangle as a function of x (say $f(x)$)

     📖 Plot the graphs of $f(x)$ and $f'(x)$ and find the maximum as we did in the previous exapmles.
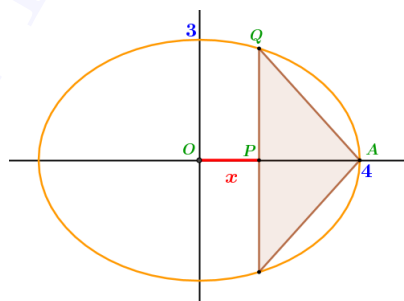
   Verification :

   - Draw the ellipse and plot the point $A(4, 0)$
   - Plot a point B on the ellipse and plot another point $C(x(B), -y(B))$
   - Using Polygon tool draw the triangle ABC and find its area.

     📖 Change the position of B and find the maximum area of the triangle.

2. Find the maximum area of an isosceles triangle inscribed in the ellipse $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$ with its vertex at one end of the minor axis. Verify your answer by constructing a GeoGebra applet.

3. Find the maximum area of an isosceles triangle inscribed in the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ with its vertex at one end of the major axis.Verify your answer by constructing a GeoGebra applet.

4. Find the maximum area of an isosceles triangle inscribed in a circle or radius 3. Verify your answer by constructing a GeoGebra applet.

Activity 31.4 Visualisation of Problems - 4

1. An open topped box is to be constructed by removing equal squares from each corner of a 3 metre by 8 metre rectangular sheet of aluminium and folding up the sides. Find the volume of the largest such box using derivatives. Verify your answer using the given applet

   Procedure

   - Taking x as the length of the squares to be removed from the corners, write the volume of the box as a function of x (say $f(x)$)
   - Draw the graph of $f(x)$ and $f'(x)$.

   📖 Find the points at which the graph of $f'(x)$ meets the x axis. Using these points find the value of x at which the volume is maximum in the domain of f. Also find the corresponding volume.

   We can Verify the answer using the applet ML31.4

   About the Applet
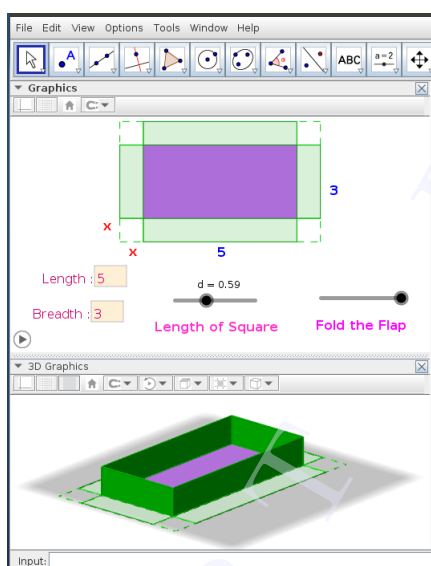
   Graphics View :- We can change the length and breadth of the rectangle using input boxes. Length of the square removing from the corners of the rectangle (x) can also be changed using a slider. We can fold the flap using another slider.

   3D Graphics :- Here we can see the box.

   - Change the length and breadth as given in the problem.

   - Change the value of x using the slider and observe the volume.

   📖 Find the maximum volume. Also find the value of x to get the maximum volume.

Activity 31.5 Visualisation of Problems -5

2. Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.

Procedure

Use Applet ML31.5

About the Applet

Radius of the cone = 2

Radius of the Cylinder = 0.5865

Curved surface area = 10.4177

R = 4

h = 2.827

Point   f(x)   f'(x)
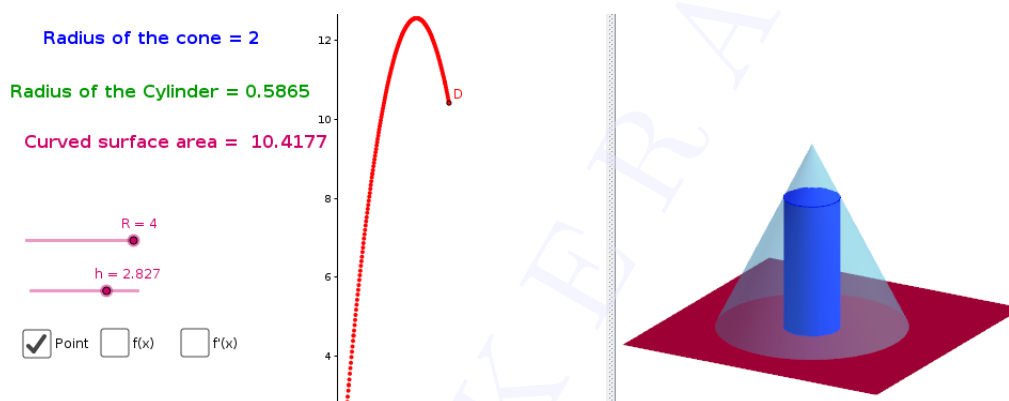
Slider 'R'	:-	To change the size of the cone
Slider 'h'	:-	To change the height of the cylinder
Check box 'Point'	:-	Shows the point D which traces the curved surface area against the radius.
Check box 'f(x)'	:-	Shows the graph of curved surface area as a function of the radius.
Check box 'f'(x)'	:-	Shows the graph of $f'(x)$.

From the Graphics View we can see the radius of the cone, radius of the cylinder and curved surface area of the cone.

- Animate the point D and observe its motion.

  Note the values of the radius of the cone and cylinder when the point D reaches the maximum height.

  Change the size of the cone, repeat the activity and verify your answer.

Lab 32

Definite Integrals

Aim

- To derive some properties of definite integrals geometrically.

- To interpret some properties of definite integrals geometrically.

- To familiarise the method of evaluating definite integral using the geometric property of curves.

Concepts

- Concept of definite integral

- Shifting and reflection of graph of a function

- Area under a curve using definite integrals

Discussion

We have already derived some properties of definite integral. In this lab we interpret this properties geometrically. We also derive some more simple properties. Some geometric properties of curves are invariant under reflection or shifting. We use this idea in our discussion. We evaluate some definite integrals using some peculiarity of graphs of the corresponding functions.

Activity 32.1 Splitting

Procedure

- Draw the graph of the function $f(x) = \sqrt{x}$

- Create three sliders **a**, **b** and **c** with Min = -10, Max = 10 and increment 0.1. Set the the sliders so that the value of **c** lies between **a** and **b** (ie, $a < c < b$)

- Using the input commands `A=Integral(f,a,b)` `B=Integral(f,a,c)` and `C=Integral(f,c,b)`. We know that A, B and C represents

$$\int_a^b f(x)dx, \int_a^c f(x)dx \text{ and } \int_b^c f(x)dx$$

respectively.

- Find the relation between A, B and C. Write this relation using definite integrals.

- Does this relation hold if the value of **c** is not in between **a** and **b**?

60
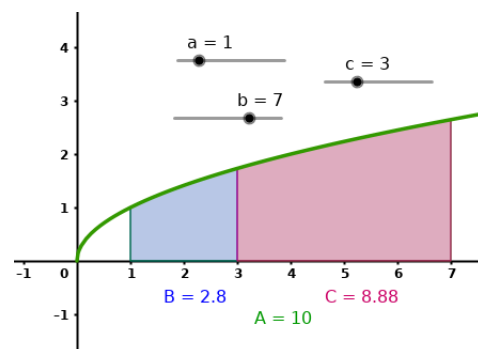
Activity 32.2 Shifting

Procedure

- Draw the graph of the function $f(x) = \sqrt{x}$

- Create a slider **c** with Min $= -10$, Max $= 10$ and increment 0.1.

- Draw the graph of $g(x) = f(x - c)$. Set $c = 0$

- Using the commands `A=Integral(f,1,4)` and `B=Integral(g,1+c,4+c)` find the area under the curves $f(x)$ and $g(x)$ between $x = 1$ and $x = 4$.

- Increase the value of **c** and observe the graph of $g(x)$ and the area of the region B. Is there any change in the area ?
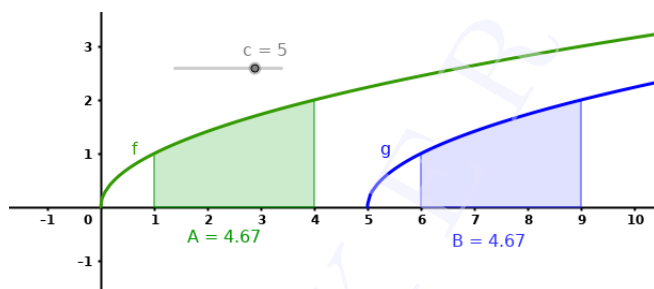
- We know that $A = \displaystyle\int_{1}^{4} f(x)dx$ and $g(x) = f(x - c)$. Write the upper limit and lower limit of B in terms of **c**. Also write B as a definite integral in terms of f and c.

- If $A = \displaystyle\int_{a}^{b} f(x)dx$, write B as a definite integral in terms of f, a, b and c

- Write your findings as a property of definite integrals.

Activity 32.3 Reflection and Shifting - 1

Procedure

- Draw the graph of the function $f(x) = \sqrt{x}$

- Using the command `A=Integral(f,1,4)` find the area under the curve $y = f(x)$ between $x = 1$ and $x = 4$.

- Draw the graph of $g(x) = f(-x)$ (graph of $g(x)$ is the reflection of the graph of $f(x)$ on y axis)

- What should be the upper and lower limits of integration so that the region under $g(x)$ is the reflection of the region under $f(x)$ on y axis? Verify your answer using integral command.(Name this definite integral as B)

- We know that area under $f(x)$ is given by $\displaystyle\int_{1}^{4} f(x)dx$ and $g(x) = f(-x)$. Write the area under $g(x)$ as a definite integral.

- If the lower and upper limits of the definite integral A are **a** and **b** respectively, write the area of both regions in terms of f, a and b

- Equating these areas, write a property of definite integrals.
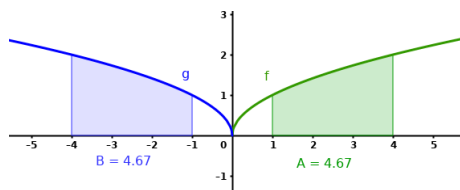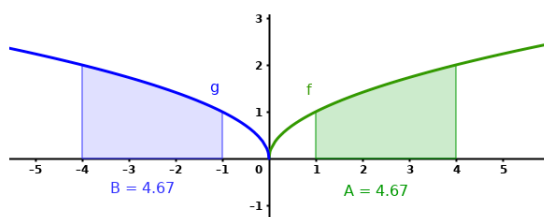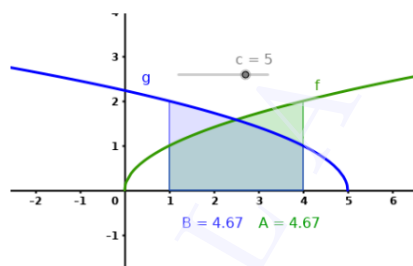
fig 1   fig 2

- What should be the horizontal shift given to $g(x)$ and the upper and lower limits so that the upper and lower limits of the region B coincides with that or region A as in fig(2).

- Let us check your guess with the help of a slider. Create a slider **c** with Min = -10, Max = 10 and increment 0.1. Set $c = 0$

- Edit the function g as $g(x) = f(-(x+c))$ and the definition of B as `B=Integral(g,-4+c,-1+c)`

- Increase the value of c until the regions coincides. How this value of c is related to the lower and upper limits of the definite integral A

- Write B as a definite integral in terms of f

- If the lower and upper limits of A are **a** and **b** respectively, write the area of both regions in terms of f, a and b

- Equating these areas, write a property of definite integrals.

- Write the property if the lower limit is 0 and upper limit is **a**.

Activity 32.4 Reflection and Shifting - 2

We have learned the property

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$$

. Let us interpret this geometrically.

Procedure

- Draw the graph of the function $f(x) = \sqrt{x}$

- Create a slider **a**.

- Using the input command `A=Integral(f,0,2a)` find the definite integral $\int_0^{2a} f(x)dx$

- We can split this region as $A = B + C$ using the input commands `B=Integral(f,0,a)` and `C=Integral(f,a,2a)`.

ie, $\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_a^{2a} f(x)dx$

Comparing this with the given property, we can see that, on the right hand side instead of

$\int_0^a f(2a-x)dx$ we have $\int_a^{2a} f(x)dx$

Let us see how they are equal

We know that $f(2a - x)$ is obtained from $f(x)$ by reflecting it on the y axis and giving a horizontal shift of $2a$ units.

- Create a slider **c** and set its value at 0.

- Create the function $g(x) = f(c - x)$, which gives the reflection of $f(x)$ on y axis.

- Since the lower and upper limits of C are a and $2a$, the lower and upper limits of the reflected region will be $-2a$ and $-a$

- Create the reflected region using the input command `D=Integral(g,-2a+c,-a+c)`
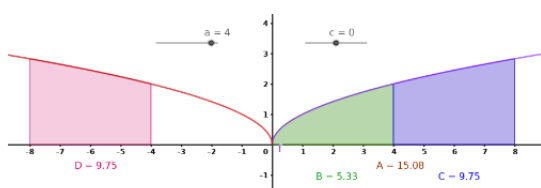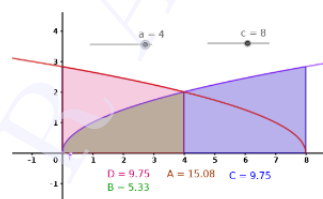
fig 1   fig 2

- Change the value of c so that the region D coincides with the region A as in fig 2.

- Compare the values of **a** and **c** and write D as a definite integral.

- In the equation $A = B + C$ replace C by D and write it using definite integrals.

Additional Activities

Activity 32.A Vertical shifting and Definite integral

- Create the function $f(x) = \sqrt{x}$

- Create a slider c and create the function $g(x) = f(x) + c$

  📖 Find $\int_1^4 g(x)dx$ for different values of c and complete the following table

c	Function	Value of Definite integral
0	f(x)	
1	$f(x) + 1$	
2		
-1		
-2		

📖 Find the relation between $\int_1^4 f(x)dx$ and $\int_1^4 (f(x) + c)dx$. Prove this relation algebraically and geometrically
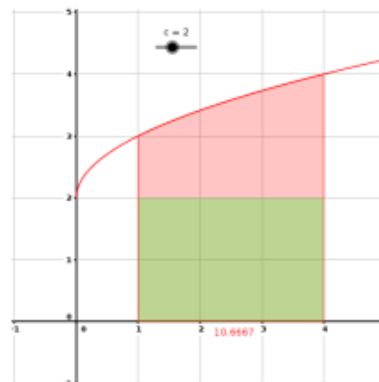
📖 In general write the relation between $\int_a^b f(x)dx$ and $\int_a^b (f(x) + c)dx$.

📖 Observe the graph of $\cos x$ and write $\int_0^{2\pi} \cos x\,dx$ and hence write $\int_0^{2\pi} (\cos x + 1)dx$.

📖 Find $\int_0^{2\pi} (\sin x + 1)dx$

Activity 32.B Definite integral and Geometry of a Curve

- Create a slider α with minimum 0 and increment 0.1
- Create the function $f(x) = \dfrac{\sin^\alpha x}{\sin^\alpha x + \cos^\alpha x}$

  📖 Draw the graph of $g(x) = f(x) - \dfrac{1}{2}$. Observe the graph for different value of α. Write $\int_0^{\frac{\pi}{2}} g(x)dx$

  📖 Find $\int_0^{\frac{\pi}{2}} \dfrac{\sin^\alpha x}{\sin^\alpha x + \cos^\alpha x}dx$

- Hide the above graphs and draw the graph of $h(x) = \dfrac{\sqrt{x}}{\sqrt{x} + \sqrt{a - x}}$

  📖 Observe the graph for different values of a and evaluate $\int_0^a \dfrac{\sqrt{x}}{\sqrt{x} + \sqrt{a - x}}dx$
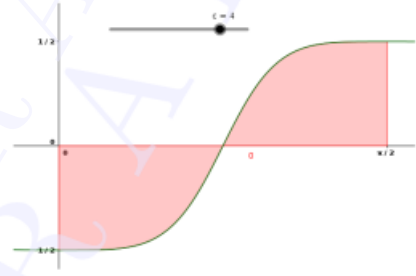
Lab 33

Definite Integrals as the limit of a sum

Aim

- To explore the concept of definite integrals as the limit of sum

- To explore the fundamental theorem of calculus

Concepts

- Definite integrals

Discussion

We discuss the definite integral as the limit of a sum .We also discuss fundamental theorem of calculus. Different cases for finding the area bounded by a curve and the x axis is also discussed

Activity 33.1 Limit of sum

Consider the function $f(x) = x^2$. Suppose we are asked to find the area under the curve bounded by the x axis and the lines $x = 1$ and $x = 3$. Since one of the boundaries of the region is a curve, which is not a straight line, we can't use the usual method of finding the area by dividing the region into triangles. Here we fill the region with rectangles and find its area as the limit of the sum of their areas.
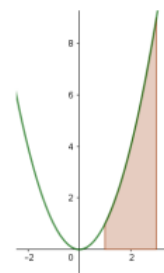
Procedure

- Draw the graph of the function $f(x) = x^2$

- Create two sliders **a** and **b**

- Create an integer slider **n** with maximum 500

- We can split the region into n rectangles of equal width using the input command `I=LeftSum(f,a,b,n)`. Set $a = 1$ and $b = 3$
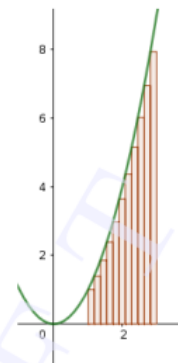
- Increase the value of n and observe what happens to the region

- The value of I, shown under the region as well as in the Algebra View gives the sum of the areas of the rectangles. Let us discuss the method of finding it.

📖 Set $n = 10$. Find the width of each rectangle

📖 Find the height of the 1st, 2nd, 3rd and the last rectangle. Also find their areas

📖 In general if the region, bounded by the curve $y = f(x)$, $x = a$, $x = b$ above the x axis divided into n rectangles of equal width, write the width h of each rectangle.

📖 If the number of rectangles is n, write the height of the 1st, 2nd, 3rd and the last rectangle

📖 Let S_n denote the sum of areas of n rectangles. Write an expression to find S_n in terms of f, a, b, h and n

- Input `h=(b-a)/n`. We can find S_n using the command `Sn=Sum(h*f(a+i*h),i,0,n-1)`. Verify that S_n and I are same.

- As we increase the value of n, we observe that the sum of areas of the rectangles approaches to the actual area under the curve. So we can find the actual area as the limit of above expression as $n \to \infty$

The limit of S_n as $n \to \infty$ is denoted by $\int_a^b f(x)dx$. We can find it using the command $Integral(f, a, b)$

📖 Find area under the curve discussed above, that is $y = x^2$, above the x axis, between $x = 1$ and $x = 3$ as the limit of sum of areas of rectangles. Verify your answer using the command `A=Integral(f,a,b)` with $a = 1$ and $b = 3$

📖 Create input boxes for **f, a** and **b**. Find area as the limit of sum of the areas of the rectangles(S_n) under the graph of the following functions between the given x values. Also find the actual area A and complete the table
1) x^2 between $x = -3$ and $x = 3$
2) x^3 between $x = 0$ and $x = 4$
3) e^x between $x = -1$ and $x = 2$
4) $\sin x$ between $x = 0$ and $x = \pi$

$f(x)$	Value of S_n for each n				Actual Area
	n=10	n=100	n=500	n=1000	
x^2					
x^3					
e^x					
$\sin x$					

Save the applet as Applet33.1

Remark In all the problems discussed above the region that we have considered lie above the x-axis. But if some part is below the x axis, $\int_a^b f(x)dx$ will not give the required area directly. Reason of this is discussed in Activity33.3. More examples of such type are given in Lab 34, Activity34.2

Activity 33.2 Fundamental Theorem of Calculus

We discuss the method of finding $\int_a^b f(x)dx$

Procedure

- Create the function $f(x) = x^2$

- Create a slider **a** and another slider **c** with Min $= -10$ and Max $= 10$. Set $a = 0$

- Using the input command `Ax=Integral(f,0,c)` find $\int_0^c f(x)dx$

  If $x = c$, `Ax` represents the area under the curve from 0 to x

- `Ax` is a function of x

- To plot the graph of `Ax`, open Graphics 2 and plot the point $A(c, Ax)$ (Input `A=(c,Ax)`). Trace the point and observe the path by animating **c**

- Using locus tool click on the slider and on the point A

- Draw the graph of $F(x) = \int f(x)dx$ on Graphics 2 using the command `F(x)=Integral(f)`

  Compare it with the graph of F(x)=Integral(f)

- Create an input box for f and change the function to $x^2 + 1$, x^3, $\sin x$ and e^x. In each case compare the graphs on Graphics 2. What do you observe?

- We can see that the graph of `Ax` either coincides with that of $F(x)$ or has a vertical shift from it. So we can write `Ax` $= F(x) + C$

📖 Write the area under the curve, from 0 to a and from 0 to b in terms of F

📖 What is the area under the curve from a to b ?

📖 Write the method of finding $\int_a^b f(x)dx$ using $\int f(x)dx$

Activity 33.3 Curve below the x axis

- Open the GeoGebra file Activity33.1 and set the function as $\sin x$ and $a = \pi$ and $b = 2\pi$

  📖 Observe that sign of I and S_n are negative. Explain the reason of this.
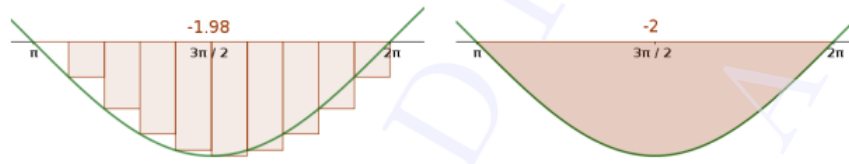
📖   How can we find the area bounded by the curve $y = \sin x$, the x axis between $x = \pi$ and $x = 2\pi$?

📖   Set $a = 0$, $b = 2\pi$ and observe the values of I and S_n. To which number do they approach as n increases ?

📖   How can we find the area bounded by the curve $y = \sin x$, the x axis between $x = 0$ and $x = 2\pi$?

Activity 33.A Area of Circle

We know that circumference of a circle of radius r is $2\pi r$. Using this we derive the area of the circle as the limiting case of areas of regular polygons inscribed in the circle.

Use applet ML33A

About the applet

- A regular polygon of number of sides n is inscribed in circle.

- We can change the number of sides of the polygon using the slider

- A triangle with one of the sides of the regular polygon as the base and the centre of the circle as a vertex is highlighted.

- h is the height of the triangle and s is the base length of the triangle. (s is the length of a side of the regular polygon).
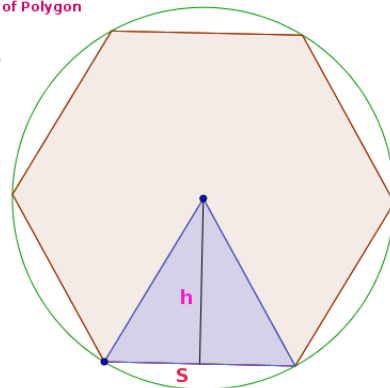
Procedure

📖   Write the area of the triangle, area of the polygon and perimeter of the polygon in terms of n, h, and s.

📖   Write the area of the polygon in terms of h and the perimeter p of the polygon.

📖   Increase the value of n and observe the polygon. What happens to h and p as $n \to \infty$?

📖   Derive the area of the circle as the limiting case of the area of the regular polygon inscribed in the circle. (Use that circumference of a circle of radius r is $2\pi r$)

Activity 33.B Volume of a Cone

Use the applet ML33B
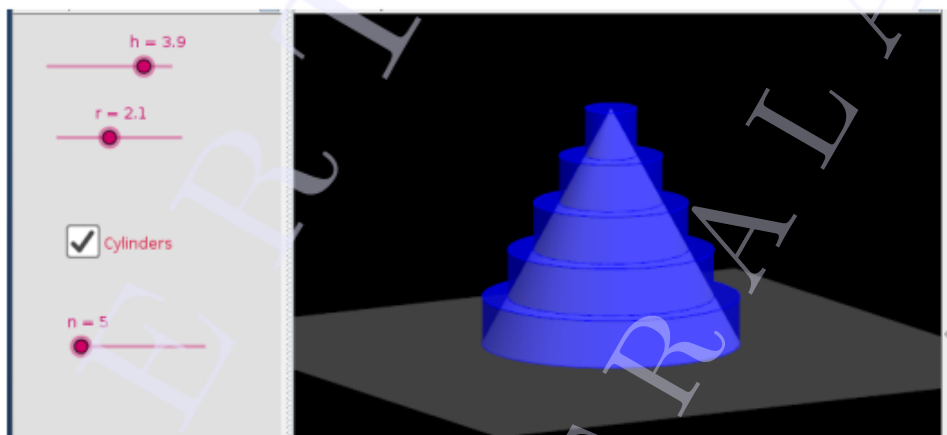
About the Applet

- Using the sliders we can change the height and radius of the cone.

- By clicking on the check box we get a pile of cylinders of the same height and a slider n

- We can increase the number of cylinders using n

Procedure

As we increase the value of n, number of cylinders increases and the pile of cylinders transforms to a cone. So the volume of the cone can be estimated as the limit of the sum of the volume of the cylinders as $n \to \infty$. An out line of the derivation is given below. Try to complete it

- Let the height of the cone is H and base radius is R

- Let the number of cylinders be n

- Then the height of each cylinder is $h = \dfrac{H}{n}$

- The radius of each cylinder is different. We have to find the radius of the ith cylinder, say r_i, from the top

- Consider the cross section

- Using the property of similar triangles we can write r_i in terms of i, h, R and H

- Find the volume of the ith cylinder

- Find the sum of volumes V_n of n cylinders

- Derive the volume of the cone as the limit of V_n as $n \to \infty$
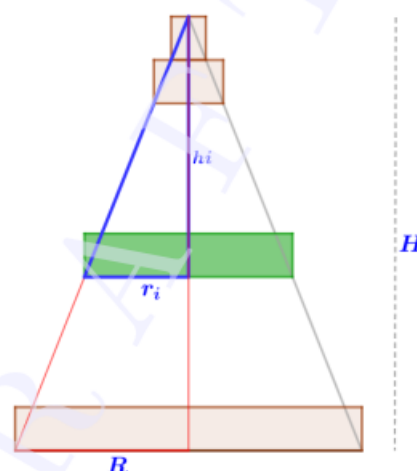
Activity 33.C Volume of a sphere

Use the applet ML33C
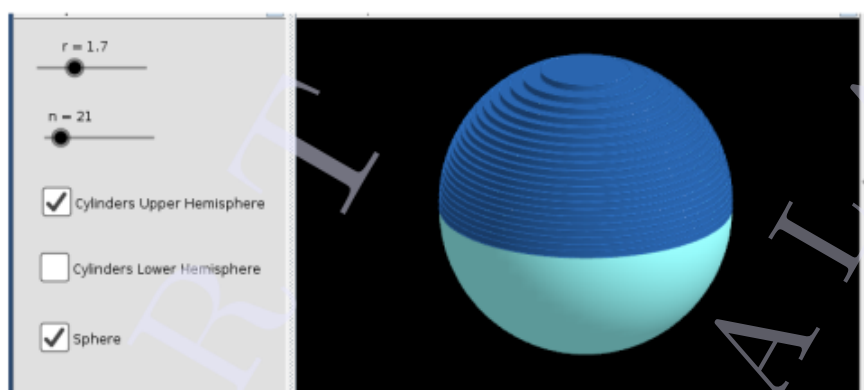
About the applet

- Using slider **r** we can change the radius of the sphere

- Using slider **n** we can change the number of cylinders

- Using the check boxes we can select the sphere, upper or lower hemisphere

Procedure

As in the above activity we can derive the volume of a sphere as the limit of sum of volumes of cylinders

An outline of the proof is given below

- Consider a hemisphere of radius R

- Height of each cylinder $h = \dfrac{R}{n}$

- Consider the cross section

- Using Pythagoras theorem we can represent the radius r_i of the ith cylinder, from the base, in terms of R, h and i

- Find the volume of v_i of the ith cylinder. (For convenience let i = 0,1,2,3,...n)

- Find the sum of volume V_n of the n cylinders

- Derive the volume of the hemisphere as the limit of V_n as $n \to \infty$

- Find the volume of the sphere
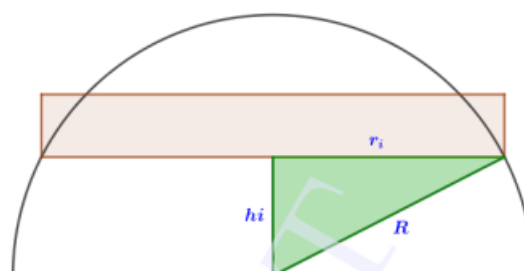
Lab 34

Application of integrals

Aim

- To find the area enclosed by curves using definite integrals

Concepts

- Definite integrals

Discussion

We discuss different methods of finding area enclosed by curves.

Activity 34.1 Area bounded by x axis

If the curve $y = f(x)$ in $[a, b]$ is above the x axis, the area of the region bounded by the curve, x axis, $x = a$ and $x = b$ is given by

$A = \int_a^b f(x)dx$. We can find this using the Input Command `Integral(f,a,b)`

📖   Find the area enclosed by the curve $y = f(x)$, x axis, $x = 0$ and $x = 2$

Procedure

- Draw the graph of the function $f(x) = x^2$.

- Find the area enclosed by the curve $y = f(x)$, x axis, $x = 0$ and $x = 2$ using the input command `a=Integral(f,0,2)`

- We can edit the lower limit and upper limit by double clicking on the region **a** (either from the Graphics view or from the Algebra view)

    📖   Find the area enclosed by the curve $y = x^2$, x axis, between

    1) $x = -1$ and $x = 3$

    2) $x = -5$ and $x = 0$

- Create an Input Box for the function $f(x)$.

    📖   Find the area enclosed by the curve $y = x^2 - 1$, x axis between $x = -1$ and $x = 1$.

    📖   Find the required area and complete the following table.

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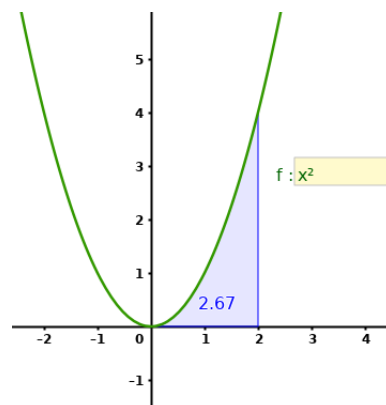
Sl.No.	Enclosed Region	Area
1	$y^2 = x, x$ axis, $x = 1, x = 4$	
2	$\frac{x^2}{16} + \frac{y^2}{9} = 1$	
3	$x^2 + y^2 = 4$	
4	$y = \sin x, x$ axis, $x = \pi, x = 2\pi$	

- save as Activity34.1

Activity 34.2 Area of a region which is above and below x axis

If a curve lies on both sides of the x axis it is not possible to find its area by direct integration. We discuss different methods of solving such problems.

Procedure

- Find the area enclosed by the curve $y = x^2 - 3, x$ axis $x = -3$ and $x = 3$

- Open the applet Activity34.1. Set $f(x) = x^2 - 3$ and edit the upper and lower limits.

What is the value of $\displaystyle\int_{-3}^{3} x^2 - 2\,dx$?. Give the reason.

What is the area of the portion below the x axis?

[To find this find the points of intersection A and B of the curve with the x axis and give the input command `Integral(f,x(A),x(B))`]

What is the area of the portion above the x axis?

Find the total area of the region.
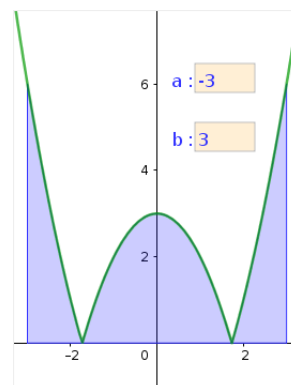
Alternate Method

- Edit the function as $f(x) = |x^2 - 3|$.

Find the area using the input command `Integral(f,-3,3)`

Find the required area and complete the following table.

Sl.No.	Enclosed Region	Area
1	$y = \sin x, x$ axis, $x = 0, x = 2\pi$	
2	$y = x^3, x$ axis, $x = -2, x = 4$	
3	$y = |x| - 2, x$ axis, $x = -1, x = 3$	
4	$y = 3x + 2, x$ axis, $x = -1, x = 1$	

Activity 34.3 Combination of curves

We discuss the methods of finding the area bounded by a combination of curves.

Find the area of the region bounded by the curve $x^2 + y^2 = 32$ and the line $y = x$.

Procedure

- Draw the circle $x^2 + y^2 = 32$ and the line $y = x$.

- Identify the enclosed region.

- Mark the point of intersection A of the line with the circle. Also mark the point of intersection B of the circle with the x axis.

- Find the area under the line $y = x$ using the input command `a=Integral(x,0,x(A))`.

- Find the area under the circle using the input command `b=Integral(sqrt(32-x^2),x(A),x(B))`.
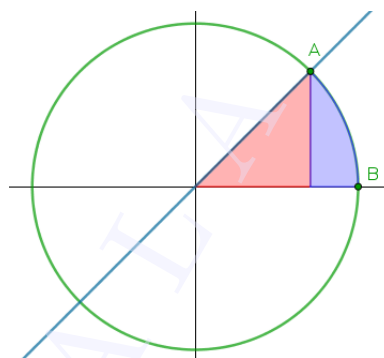
  📖 Find the required area.

Alternate method

- Define the function as $f(x) = \begin{cases} x & \text{if } 0 < x < x(A) \\ \sqrt{32 - x^2} & \text{if } x(A) < x < x(B) \end{cases}$

  For this give the input command `f(x)=If(0<x<x(A),x,x(A)<x<x(B),sqrt(32-x^2)`

  📖 Find the area using the input command `Integral(f,0,x(B))`

  📖 Find the area of the following regions.

1) Region in the first quadrant bounded by x axis, the line $y = \sqrt{3}x$ and the circle $x^2 + y^2 = 4$

2) Region lying above x axis and included between the circle $x^2 + y^2 = 8x$ and inside of the parabola $y^2 = 4x$.

3) Region enclosed between the two circles $x^2 + y^2 = 4$ and $(x - 2)^2 + y^2 = 4$.

4) The Region $\{(x, y) : 0 \le y \le x^2, 0 \le y \le x + 1, 0 \le x \le 2\}$

Activity 34.A Availability of medicine in the blood

Medicine can be administered to patients in different ways. For a given method let c(t) be the concentration of the medicine in the blood t hours after the dose is given. Over the time interval $0 \le t \le a$ the area between the grapf of $c(t)$ in the interval $[0, a]$ indicates the availability of the medicine for the patients body over the time.

Since GeoGebra does not have the time axis, we will change the independent variable to x.

Two methods Method 1 and Method II for administrating the medicine is governed by the functions $f(x)$ and $g(x)$ represented as follows

$$f(x) = 5(e^{-0.2x} - e^{-x})$$

$$g(x) = 4(e^{-0.2x} - e^{-3x})$$

In this activity we try to explore the availability of medicine to the patients body

Procedure

- Create a slider **a** with minimum value 0 and maximum value 24 , corresponding to 24 hour period

- Create the function $f(x)$

- Create the function $g(x)$

- Evaluate the integral of f from 0 to **a** using integral function .This can be achieved by inputting `Integral(f,0,a)`

- Evaluate the integral of g from 0 to **a** as above .In the algebra view we can see that two variables are created for the area of the two figures

- Move the slider **a** and observe the change in the areas

If 4 is the value of **a** , which area is more? Which method provides better availability of the medicine?

If 24 is the value of **a**, which area is more? Which method provides better availability of the medicine?

Move the slider in such a way that the areas of both the functions are same. What does this condition signify in medical perspective?

Lab 35

Application of integrals 2

Aim

- To find area enclosed by curves and geometrical figures by definite integrals

Concepts

- Definite integrals

Discussion

We find area bounded by two curves and area of some geometrical figures by definite integrals.

Activity 35.1 Area between two curves

📖   Find the area of the region bounded by the parabolas $y = x^2$ and $y^2 = x$

Procedure

- Draw the curves using suitable inputs.

- Find the intersecting points A and B of the curves using the `Intersect` tool.

    📖   Identify the functions $f(x)$ and $g(x)$ which forms the boundaries of the region.

    📖   Find the area under the curve $y = f(x)$ between $x = x(A)$ and $x = x(B)$ using the input command `a=Integral(f,x(A),x(B))`
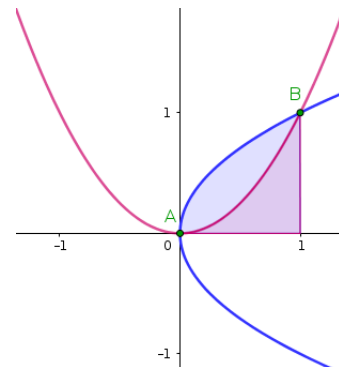
📖   Find the area under the curve $y = g(x)$ between $x = x(A)$ and $x = x(B)$ using the input command `b=Integral(g,x(A),x(B))`

📖   Using above values find the area enclosed by the curves.

Alternate method

We can find the area enclosed between the curves using the Input Command
`IntegralBetween(sqrt(x),x^2,x(A),x(B))`

📖 Find the area of the following regions.

1) Region in the first quadrant enclosed by the curves $9x^2 + y^2 = 36$ and the line $\dfrac{x}{2} + \dfrac{y}{6} = 1$

2) Region bounded by the curves $y = x^2 + 2$, $y = x, x = 0$ and $x = 3$.

3) Region bounded by the curve $|x| + |y| = 1$.

4) Region bounded by the parabola $y = x^2$ and the line $y = x$.

5) Region bounded by the y axis, $y = \cos x$ and $y = \sin x$ when $0 \leq x \leq \dfrac{\pi}{2}$.

6) Region between the ellipse $\dfrac{x^2}{36} + \dfrac{y^2}{9} = 1$ and the circle $x^2 + y^2 = 9$.

Activity 35.2 Area enclosed by y axis

📖 Find the area of the region enclosed by the curve $y = x^2$, y axis , between $y = 1$ and $y = 3$ in the first quadrant.

Procedure

- Create a function the function $f(x) = x^2$ and identify the region.

- Find the inverse function of f by giving the input `g(x) = Invert(f)`

  📖 Identify the region under the curve $y = g(x)$ which has the same area of the given region and find the required area
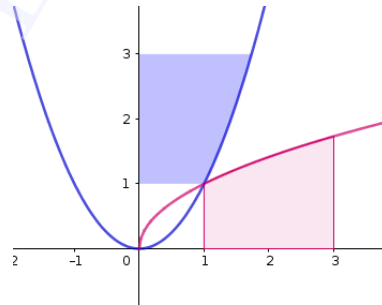
📖 Find the area of the following regions.

1) Region bounded by $y^2 = 4x, y = 2, y = 4$ and the y axis in the first quadrant.

2) Region lying in the first quadrant and bounded by $y = 4x^2, y = 1$ and $y = 4$.

3) Region bounded by $y = \sin^{-1} x$, $y = 0$, $y = \dfrac{\pi}{2}$.

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Activity 35.3 Area of Geometrical figures

Find the area iof the given figures.

1.

Curves used are parabolas $y = \pm x^2$ and $x = \pm y^2$

2.

Curves used are circles of radius 2 centered at $(0,0)$ and $(1,0)$

3.
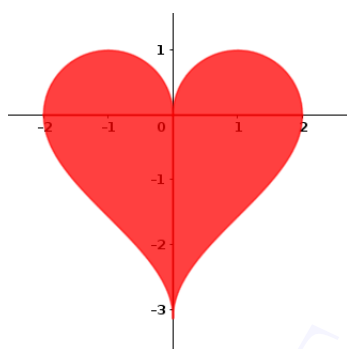
Curves used are semi circles of radius 1 centered at $(-1,0)$ and $(1,0)$, shifting of the curve $y = \sin^{-1} x$ and its reflection on y axis.

4.
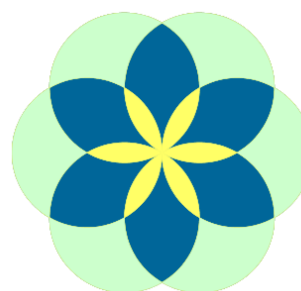
Curves used are circles of radius 2. Find the area of the yellow region, blue region and green region. Also find the total area of the figure.

Activity 35.A Area between curves with a constant difference

In this activity we explore the area between the two curves $f(x)$ and $f(x) + c$ in the interval $[a, b]$

Procedure

- Create 3 sliders **a,b** and **c**

- Create the function $f(x) = x^2$

- Create the function $g(x) = x^2 + c$

- Evaluate the area between the two curves f and g between a and b

For different values of **a , b, c** as shown in the table below, note down the area between the curves

a	b	c	Area
0	1	2	
0	2	2	
0	3	2	
0	4	2	
1	2	3	
1	3	3	
1	4	3	

Repeat the activity for different functions x^3 and $2x$

📖   Can we draw a common inference from this acivity?

📖   Can you mathematically prove your result?

Lab 36

Differential Equations

Aim

- To explore the concept of differential equations geometrically
- To explore the geometrical meaning of general and particular solutions of differential equations.
- To explore the significance of the constant in the general solution of a DE

Concepts

- Concept of differential equation
- Order of a DE
- Slope of tangent to a curve is the derivative at that point
- General and particular solutions of differential equations

Discussion

We represent a family of curves with the help of differential equations. We also discuss the geometrical meaning of general and particular solutions of differential equations.

Activity 36.1 Differential Equation of a family of curves

1. Find geometrically the DE representing the family of curves $y = x^2 + c$

   Procedure

   - Create a slider **c** and the function $f(x) = x^2 + c$.
   - Create another slider **a** and plot the point $A(a, f(a))$.
   - Draw the tangent to the curve at A and find its slope.
   - Show the coordinates of A.
   - Set $c = 0$.

     📖 Observe the coordinates of A and the slope of the tangent. Find the relation between them. Do this for different values of a.

     📖 Write this relation using $\dfrac{dy}{dx}$.

     📖 Change the value of c and observe whether the same relation holds.
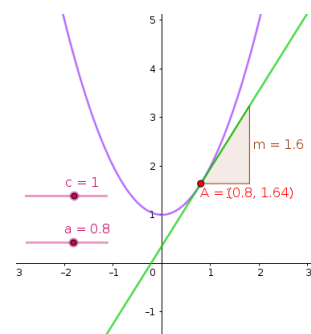
- Above equation in $\dfrac{dy}{dx}$ is called the differential equation representing the family of curves $y = x^2 + c$. We get the same equation algebraically by eliminating c using differentiation.

2. Find, geometrically, the DE representing the family of curves $y = ce^x - x$

3. Find, geometrically, the DE representing the family of curves $y = ce^x - 2x - 2$

Activity 36.2 Second order DE

Find geometrically the DE representing the family of curves $y = a\sin x + b\cos x$.

Procedure

- Create two sliders **a** and **b**.
- Draw the graph of the function $f(x) = a\sin x + b\cos x$.
- Draw the graph of $f'(x)$.
- Create a slider **c** and plot the points $A(c, f(c))$ and $B(c, f'(c))$.
- Draw the tangent at B and find its slope.
- Show the coordinates of A.

  📖 Observe the coordinates of A and the slope of the tangent and find the relation.
- Do this for different values of a and b.
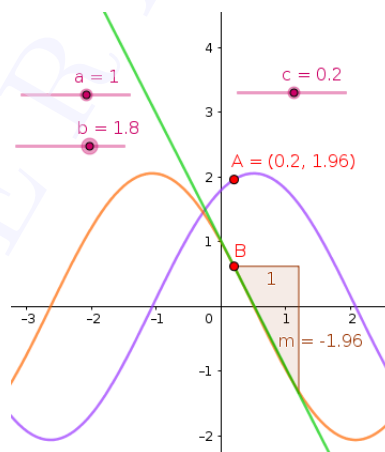
  📖 Write the differential equation representing the family of curves $y = a\sin x + b\cos x$.

Activity 36.3 Solution of DE

We can use the `SolveODE` command to find general and particular solutions of differential equations. If the DE is of the form $\dfrac{dy}{dx} = f(x, y)$, give the command as `SolveODE(f(x,y))` . If the solution satisfies the condition $y = y_1$ when $x = x_1$ then the particular solution is obtained by the Input Command `SolveODE(f(x,y),(x_1,y_1))`

Let us consider some examples

1. To find the general solution of the DE $\dfrac{dy}{dx} = 2x$, give the input command `SolveODE(2x)`. In the algebra view we get the solution as $y = x^2$ where the general solution is $y = x^2 + c$. instead of the constant c we get a number c_1 in the algebra view. Clicking on it we get a slider in the graphics view. Corresponding graph is also shown in the graphics view.

2. Solve the DE $\dfrac{dy}{dx} = \dfrac{x}{y}$. Change the value of the slider c_1 and observe the graph. What happens to the curve when $c_1 = 0$? Why ?

3. Find the equation of a curve passing through the point (1,2) given that the slope of the tangent to the curve at any point is the sum of the coordinates at that point. Verify your answer using GeoGebra.

Procedure

- It is easy to see that the DE corresponding to the given conditions is $\frac{dy}{dx} = x + y, y = 0$ when $x = 0$.

- Solution of the DE is obtained by the command
  `SolveODE(x+y,(1,2))`

  and its graph will be shown in the Graphics view (named as f).

- Create a slider **a** with increment 0.1 and plot a point $A(a, f(a))$ on the curve. Show the coordinates of A.

- Draw the tangent to the curve at above point and find its slope.

  📖 Change the value of **a** and check whether the slope of the tangent is same as the sum of coordinates of A
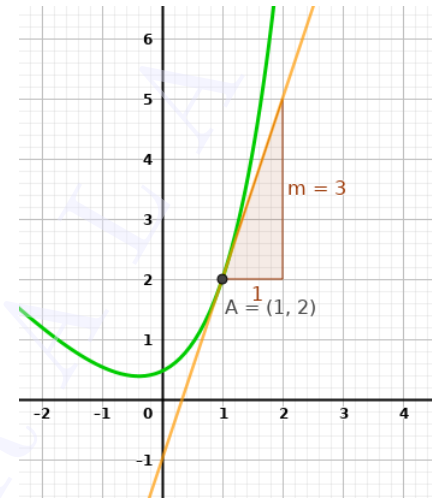
4. Let us discuss the significance of the constant in the general solution of a DE

   - Find the solutionof the DE $\frac{dy}{dx} = x - y$ using the input command `SolveODE(x-y)`.
   - Change the value of the constant c_1 and observe the curve. For what value of c_1 it becomes a straight line.
   - Write a differential equation of above line .

Additional Activities

Activity 36.A Logistic Model

Even though uninhibited growth is the most simplest model to describe population growth, it is not realistic since the assumptions of infinite resources is not to be found in nature. An ecosystem has a maximum carrying capacity, say L. If the population is below the carrying capacity, then the rate of population growth is positive and population increases. But if the population is greater than the carrying capacity, then the population decreases and the rate is negative. A differential equation that meets all the requirements discussed above is as follows

$$\frac{dy}{dt} = k \left(1 - \frac{y}{L}\right) y$$

where k is the constant of proportionality.

Procedure

- Create a slider k for the proportionality constant with a minimum value of -15 to maximum value of 15 with increment 0.01

- Create a slider L for the maximum holding capacity with minimum value of 0 to maximum value 50.

- Using the SolveODE command find the solution of the differential equation by entering the following command in the input box $SolveODE(k * (1 - y/L) * y)$

- Set the value of k to 3, L to 5 and the constant c_1 to -5.

  📖 To observe the derivative of the function $f(x)$, enter the following command in the input bar $Derivative(f)$

To check whether the solution corresponds to the original definition enter the following command in the input bar $k * (1 - f/L) * f$.

Can you come up with a physical explanation for the graph. f is the population and f' is the rate of population rise.

By adjusting the values of sliders, can you obtain a constant solution for the problem? Is there only one constant solution, or is there more than one constant solutions. Can you mathematically give a reason for this?

Activity 36.B Linear and Non linear Differential Equations

In this activity we try to explore the difference between a linear differential equation and a non linear differential equation.

Procedure

- Use SolveODE command to find the solution of the differential equation $\frac{dy}{dx} = y$

- An auxillary object c_1 is created, which is a slider and can be visible if the show option for the variable is enabled.

- Using the Point on Object tool, create a point on the function f. The point created is A.

- Using the Tangent tool create a tangent at this point for the function f.

- Using the slope tool, find the slope of the tangent. A slope variable m is created.

- Create a new point with the following command $(y(A), m)$. Trace on for this point.

- Right click on the point A and set the animation on.

    Note the shape of the trace. What does this represent.

    Change the value of c_1 and observe what happens to the trace of the point.

    Can you provide a theoretical explanation for this behaviour.

- Repeat the same process for the following equations as well.

$$\frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = y^2$$

Among the three functions can you differentiate which functions are linear and which functions are non linear.

Activity 36.C Drug Delivery

When a drug enters the human body, it enters the bloodstream and eventually will be absorbed by the body. It has been observed that the amount of drug in the blood decreases at a rate that is proportional to the drug present at a time. This can be modeled as a differential equation

$$\frac{dy}{dt} = -ky$$

Procedure

- Create a slider k with a minimum value of 0 and maximum value of 15

- In the input bar use $SolveODE(-k*y)$

- Make the auxillary slider c_1 visible

- Set the value of k as 1 and c_1 as 1.

- Half life is defined as the time taken for the drug to reduce to half its initial value. To see the half life of the drug, we do the following. Draw a line y=0.5. A line g is created.

- Intersect the curve f and the line g. A point A is created.

  Write the x coordinate of A. This is the half life of the drug.

  Adjust the value of the slider k to see the change in the half life. When k increases, what happens to half life of the drug?

  The half life of the famous analgesic, Paracetamol is found to be 1-4 hours. If we assume that the half life is 3 hours, can you find the proportionality constant k for Paracetamol?

  Change the value of the slider c_1 and observe the change in the point A. We normally do not relate the constant c_1 to half life. Can you see why it is so? What is the physical meaning of c_1?

Lab 37

Vectors

Aim

- To explore the concept of vectors and vector operations

- To explore the concepts direction angles and direction cosines

- To resolve a vector in given directions

Concepts

- vectors and vector operations

- Direction angles and direction cosines of a vector

- Components of a vector

Discussion

We familiarise some tools and commands for the construction of vectors and vector operations in GeoGebra, using which we discuss some problems on vectors which helps to get the geometry of vectors.Discuss direction angles and direction cosines of vectors geometrically.We also discuss the method of resolving a vector in given directions .

Tools and Commands

Let's familiarise some tools and commands used for the construction of vectors and vector operations in GeoGebra.

- `Vector` tool or the Input Command `Vector(Point,Point)` gives the vector which starts from the first point and ends at the second point.

- The Input Command `Vector(Point)` gives the vector starts from the origin and ends at the point.

- `Vector from Point` tool, by clicking on a point and on a vector gives the vector starts from the point and parallel to the given vector.

- The Input Command `Point(Point,Vector)` Gives the end point of the given vector which starts from the given point.

- `Translate by Vector` tool, by clicking on an object translates it by the magnitude of the vector in its direction.

- `|u|` gives the magnitude of $\vec{u}$

- If a is a scalar and u is a vector, then the Input command `au` or `a*u` gives the vector $a\vec{u}$

- If u and v are vectors, the Input command `u+v` gives $\vec{u} + \vec{v}$ and `u-v` gives $\vec{u} - \vec{v}$.

📖 Do the following problems

Activity 37.1 Construction of Vectors

1. Construct the vector $3\hat{i} + \hat{j} + 2\hat{k}$.

2. Construct the vector $3\hat{i} + \hat{j} + 2\hat{k}$ starting from the point $(2, -2, 3)$.

3. Construct the vector joining the points $(1, -2, 4)$ and $(3, 2, -2)$.

4. Construct three mutually perpendicular vectors with same initial point $(2, 2, 2)$.

5. Construct the parallelogram whose adjacent sides are represented by the vectors $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{b} = \hat{i} - 4\hat{j} - 3\hat{k}$

6. Create a slider **a** and a vector $\vec{u}$. Construct the vector $a\vec{u}$. Observe the vector by changing the value of a.

7. Construct the vector $\vec{u} = 4\hat{i} - 3\hat{j} + 3\hat{k}$. Construct the unit vector in the direction of $\vec{u}$. Also Construct the vector having magnitude 5 in the direction of $\vec{u}$.
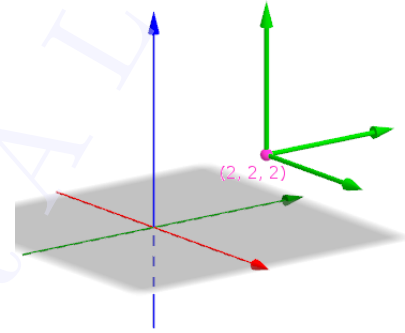
Activity 37.2 Direction ratios and Direction Cosines

Procedure

- Plot the points $O(0, 0, 0)$, $A(1, 0, 0)$, $B(0, 1, 0)$, $C(0, 0, 1)$ and $P(4, 3, 2)$.

- Construct the vector $\overrightarrow{OP}$.

- Find the angles α, β and γ made by $\overrightarrow{OP}$ with the positive direction of x axis, y axis and z axis. [input : α=Angle(A,P), β=Angle(B,P),γ=Angle(C,P)]. Hide the points A, B and C.

- Draw the perpendicular from P to the x axis and plot its point of intersection D with x axis.

  📖 Write the x coordinate of D.

  📖 Find $\cos\alpha$ without using the input command $\cos(\alpha)$. Similarly find $\cos\beta$ and $\cos\gamma$.

  📖 If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ write $l = \cos\alpha, m = \cos\beta, n = \cos\gamma$ in terms of x, y, z.

  📖 Prove that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$
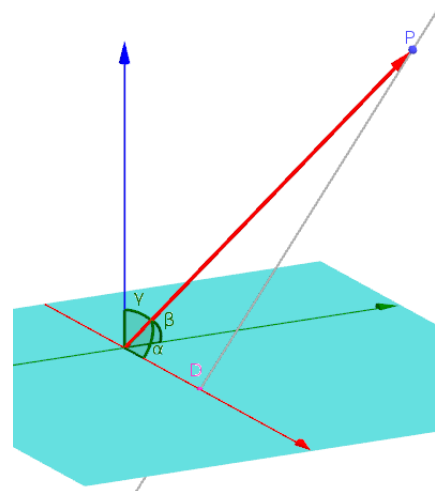
Open Graphics View and create an input box for the point P(It is not possible to create input box in 3D Graphics)

📖 Edit the co-ordinates of the point P such that it lies on the x- axis. What happens to the direction cosines? What happens if P lies on the y-axis or z-axis?

📖 Edit the co-ordinates of P such that it lies on the xy plane . What happens to n? Write l and m in terms of α only.

Activity 37.3 Components of a vector

Procedure

Construct the components vectors of the vector $\vec{r} = 2\hat{i} + 5\hat{j} - 3\hat{k}$ along the positive direction of the coordinate axes.

- Plot the point $A(2, 5, -3)$ and construct the vector $\vec{r}$ using the input command `r=Vector(A)`.

- Plot the points $B(x(A), 0, 0), C(0, y(A), 0)$ and $D(0, 0, z(A))$. Draw the segments AB, AC and AD.

- Construct the component vectors.

  📖 Create an input box for the point A. Write the vector $\vec{r}$ and its component vectors for the following positions of A, $(3, 2, 0), (0, 5, -2), (0, 4, 0), (0, 0, -3)$

Express the vector $\vec{r} = 4\hat{i} + 4\hat{j}$ as the sum of two vectors which are parallel to $\vec{u} = 3\hat{i} + \hat{j}$ and $\vec{v} = \hat{i} + 6\hat{j}$

- Construct $\vec{r}$ as in the last discussion
- Construct the vectors $\vec{u}$ and $\vec{v}$.
- Draw the lines passing through the origin and parallel to the vectors $\vec{u}$ and $\vec{v}$. For this give the input commands `Line((0,0,0),u)` and `Line((0,0,0),v)`
- Draw the line passing through the point A (end point of $\vec{r}$) and parallel to $\vec{v}$. Plot its point of intersection B with the line parallel to $\vec{u}$
- Draw the vectors $\overrightarrow{OB}$ and $\overrightarrow{BA}$ where O is the origin.
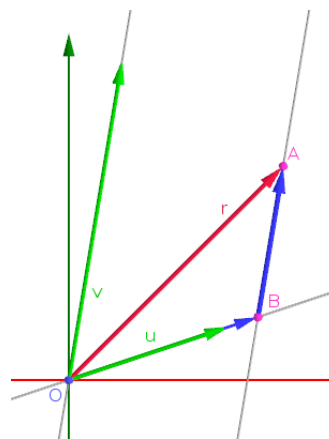
Additional Activities

Activity 37.A Components of a vector in space

Express the vector $\vec{r} = 2\hat{i} + 5\hat{j} + 3\hat{k}$ as the sum of the vectors parallel to $\vec{a} = 4\hat{i} + 2\hat{j}$, $\vec{b} = -\hat{i} + 4\hat{j}$ and $\vec{c} = \hat{i} + \hat{j} + 5\hat{k}$

- Construct the points $P(2,5,3)$, $A(4,2,0)$, $B(-1,4,0)$ and $C(1,1,5)$ hence construct the vectors $\vec{r}, \vec{a}, \vec{b}$ and $\vec{c}$.

- Draw the line passing through A and parallel to $\vec{c}$ and plot its point of intersection D with the xy plane. (xy plane is available in the 3D view by default. Otherwise give the input command z=0.

- Draw the line passing through O and parallel to $\vec{a}$

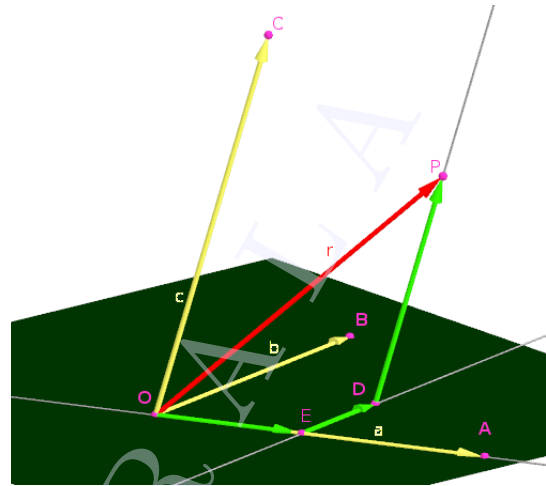- Draw the line passing through D and parallel to $\vec{b}$ and plot its point of intersection E with the line constructed in the last step.

- Draw the vectors $\overrightarrow{OE}, \overrightarrow{ED}$ and $\overrightarrow{DP}$.

Can you express the vector $\vec{r} = 2\hat{i} + 5\hat{j} + 3\hat{k}$ as the sum of the vectors parallel to $\vec{a} = 4\hat{i} + 2\hat{j}$, $\vec{b} = -\hat{i} + 4\hat{j}$ and $\vec{c} = \hat{i} + \hat{j}$? Why ?

Activity 37.B The Vector Clock

In this activity a clock is viewed from the perspective of vectors. The origin is assumed as the center of the clock and vectors are drawn from the origin to each hour position.

procedure

- Create a point A(0,0)

- Create a point B(0,5)

- Using the Sequence command, create a list of points which are a result of rotation of B by an angle $30^0, 60^0, 90^0, \ldots 330^0$. This can be achieved by the command $Sequence(Rotate(B, -k * (360/12)^0), k, 0, 11)$. A list $l1$ is created.

- Draw vectors from A to the list created in the previous step by using Sequence command. This can be achieved by $Sequence(Vector(A, l1(k)), k, 1, 12)$. A list of vectors $l2$ is created.

- Translate the vector $l2(2)$ to the tip of $l2(1$ using the command $Translate(l2(2), l2(1))$. A point C is created.

- Draw a vector from the tip of $l2(1)$ to the point C by using the command $Vector(l2(1), C)$

- Translate the vector $l2(3)$ to the tip of C using the command $Translate(l2(3), C)$. A point D is created.

- Draw a vector from the tip of C to the point D by using the command $Vector(C, D)$

- Translate the vector $l2(4)$ to the tip of D using the command $Translate(l2(4), D)$. A point E is created.

- Draw a vector from the tip of D to the point D by using the command $Vector(D, E)$

- Continue this for all the vectors which represent the hands of the clock.

  📖 What is the sum of the vectors of the hands of the clock?

👉

The sum of the vectors can be achieved using a single command Sum. In the input bar type $Sum(l2)$. This will create a vector whose sum is the vector sum of all the vectors in the list $l2$.

What is the sum of all the vectors if the vector corresponding to 2:00 is dropped off? Use the command Sum by including all the items in the list *l2* except the one corresponding to 2:00. This can be achieved by the command

$$Sum(l2(1) + l2(2) + l2(4) + l2(5) + l2(6) + l2(7) + l2(8) + l2(9) + l2(10) + l2(11) + l2(12))$$

What if the 1:00 hour handle is dropped?

Can you generalise this activity?

If two consecutive vectors are removed, say the vector corresponding to 1:00 and 2:00 are removed, what will happen to the resultant?

If the vectors corresponding to 1:00 and 3:00 are removed, what is the resultant?

If all the twelve vectors start from (0,-5) corresponding to 6:00, what will the resultant be? To do this in the algebra view right click on the point A and in the object properties change the coordinates from (0,0) to (0,-5)

Lab 38

Vectors 2

Aim

- To explore the concepts of dot product, cross product and scalar triple product geometrically.

Concepts

- Dot product

- Cross product

- Scalar triple product

Discussion

We do some problems involving the concept of dot product, cross product and scalar triple product and verifies it geometrically

Activity 38.1 Dot Product

The input command **u*v** or **Dot(u,v)** gives $\vec{u} \cdot \vec{v}$

📖   Do the following problems

1. Show that the vectors $\vec{a} = 3\hat{i} - 2\hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} - 4\hat{k}$ are perpendicular to each other. Draw the vectors and verify your answer.

2. Find the angle between the vectors $\vec{a} = \hat{i} + \hat{j} - \hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + \hat{k}$. Draw the vectors and verify your answer.

3. Find the projection of the vector $\vec{a} = \hat{i} + 3\hat{j} + 7\hat{k}$ on the vector $\vec{b} = 7\hat{i} - \hat{j} + 8\hat{k}$. Verify your answer using GeoGebra.

   Procedure

   - Plot the points $O(0,0,0)$, $A(1,3,7)$ and $B(7,-1,8)$
   - Construct the vectors $\vec{a} = \overrightarrow{OA}$ and $\vec{b} = \overrightarrow{OB}$
   - Find $\vec{a} \cdot \vec{b}$, $|\vec{b}|$ and hence find the projection of $\vec{a}$ on $\vec{b}$

   Verify the answer as follows.
   - Draw the line OB
   - Draw the perpendicular line from A to the line OB and plot their point of intersection C.
   - Find the distance OC

4. Show that the vectors $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{b} = \hat{i} - 3\hat{j} - 5\hat{k}$ and $\vec{c} = 3\hat{i} - 4\hat{j} - 4\hat{k}$ form the vertices of a right angled triangle. Construct the triangle and verify.

   Procedure

   - Plot the points A, B and C whose position vectors are $\vec{a}, \vec{b}$ and $\vec{c}$ respectively.

   - Draw the vectors representing the sides of the triangle.

   - Using dot product show that the triangle is a right angled triangle.

   - Construct the triangle using `Polygon` tool, find its angles and verify that it is a right angled triangle.

5. Construct the vector $\vec{u} = 2\hat{i} + 3\hat{j} + 3\hat{k}$.

   - Construct a plane perpendicular to $\vec{u}$. [Plot any point - say $A(0, 0, 0)$ and give the input command `PerpendicularPlane(A,u)`].

   - Plot any two points B and C on the plane and draw the vector $\vec{v} = \overrightarrow{BC}$

   - Find $\vec{u} \cdot \vec{v}$. Change $\vec{v}$ by changing the position of B or C. What happens to $\vec{u} \cdot \vec{v}$? Why?
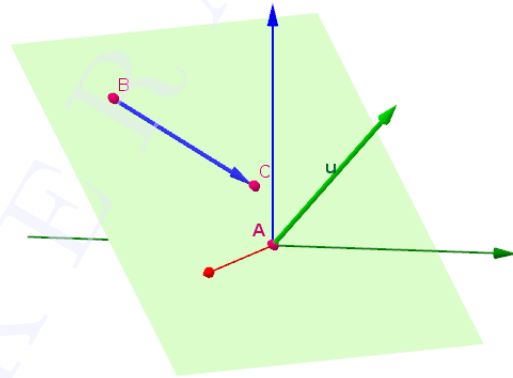
Activity 38.2 Cross product

The input command `u⊗v` or `Cross(u,v)` gives $\vec{u} \times \vec{v}$

📖   Do the following problems

1. Consider the vectors $\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + \hat{j}$.

   - Construct $\vec{a} \times \vec{b}$

   - By marking the angles verify that $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$

   - Find $\vec{b} \times \vec{a}$ and compare it with $\vec{a} \times \vec{b}$.

   - Find a unit vector $\hat{n}$ such that $\vec{a}, \vec{b}, \hat{n}$ form a left handed system of vectors.

2. Consider the points $A(1, 1, 1), B(3, 3, 4)$ and $C(-2, 3, 3)$

   - Construct a vector perpendicular to the plane containing the points A, B and C. [Input Command `Plane(A,B,C)`].

   - Construct a vector of magnitude 5 which is perpendicular to the plane containing the points A, B and C.

3. Construct the parallelogram whose adjacent sides are represented by the vectors $\vec{a} = 3\hat{i} + \hat{j} + 4\hat{k}$ and $\vec{b} = \hat{i} - \hat{j} + \hat{k}$. Find $\vec{a} \times \vec{b}$ and hence find the area of the parallelogram. Verify your answer using `area` tool.

4. Construct three mutually perpendicular vectors, with initial point $(2, 2, 2)$, which are not parallel to the coordinate axes.
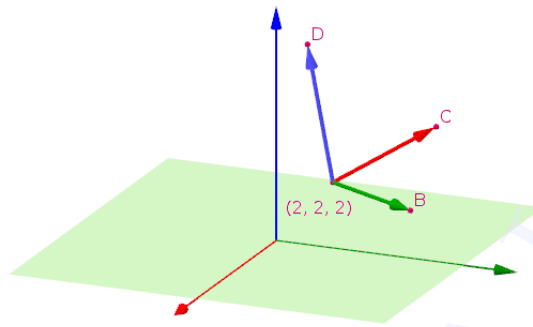
Activity 38.3 Triple Product

The input command `a*(b⊗c)` or `Dot(a,Cross(b,c))` gives the value of $\vec{a} \cdot (\vec{b} \times \vec{c})$.

📖 Do the following problems.

1. Take three non zero non parallel vectors $\vec{a}, \vec{b}$, and $\vec{c}$. Find $[\vec{a}, \vec{b}, \vec{c}]$, $[\vec{b}, \vec{c}, \vec{a}]$, $[\vec{c}, \vec{a}, \vec{b}]$, $[\vec{a}, \vec{c}, \vec{b}]$, $[\vec{c}, \vec{b}, \vec{a}]$ and $[\vec{b}, \vec{a}, \vec{c}]$ and compare their values.

2. If three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are given, how can we find whether they are in right handed system or not ?

   • Construct the vectors $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ (Give the input commands `a=Vector((1,1,2))` and `b=Vector((-1,2,1))`)

   • Construct $\vec{a} \times \vec{b}$

   • Construct the plane determined by $\vec{a}$ and $\vec{b}$. [Input command `Plane((0,0,0),a,b)`]

   • Plot the point $A(-2, 1, 2)$ and construct $\vec{c}$ using the input command `c=Vector(A)`

   • Say whether $\vec{a}, \vec{b}, \vec{c}$ form a right handed system or not
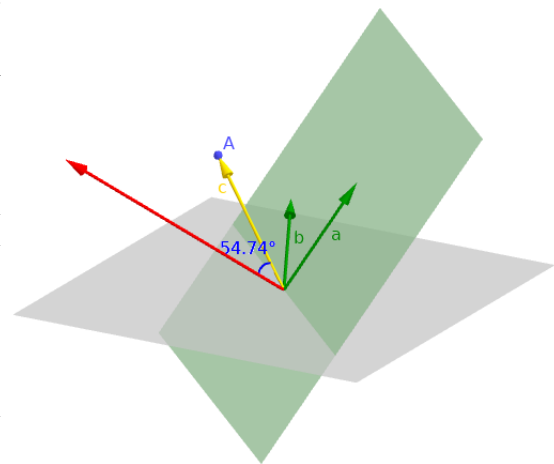
   • Find the angle between $\vec{c}$ and $\vec{a} \times \vec{b}$

   • Find $[\vec{a}, \vec{b}, \vec{c}]$ and observe its sign.

   • Click and drag the point A towards the plane, observe that the angle increases. What happens to the angle when the point reaches the plane. What is the value of $[\vec{a}, \vec{b}, \vec{c}]$ at this stage ?

   • Drag the point again, so that the angle increases again.(Then $\vec{a} \times \vec{b}$ and $\vec{c}$ are on opposite sides of the plane).

   • Say whether $\vec{a}, \vec{b}, \vec{c}$ form a right handed system or not. Observe the sign of $[\vec{a}, \vec{b}, \vec{c}]$. Why it happens so.

   📖 How can we determine whether three vectors are in right handed system, left handed system or coplanar from the sign of $[\vec{a}, \vec{b}, \vec{c}]$?

📖 How the angle between the vectors $\vec{a} \times \vec{b}$ and the sign of $[\vec{a}, \vec{b}, \vec{c}]$ are related ? Why ?

3. Consider the vectors $\vec{a} = 2\hat{i} + 4\hat{j} + \hat{k}$ and $\vec{b} = -3\hat{i} + 3\hat{j} + 2\hat{k}$.

   - Construct the parallelogram represented by $\vec{a}$ and $\vec{b}$ and find its area.
   - Construct the parallelepiped of volume 50 and having above parallelogram as its base.
     - If a is the area of the parallelogram, its height will be $h = \dfrac{50}{a}$
     - Construct $\vec{a} \times \vec{b}$ and the vector $\vec{c}$ having magnitude h in the direction of $\vec{a} \times \vec{b}$.
     - Find the end point C of $\vec{c}$.
     - Construct the plane passing through C and perpendicular to $\vec{c}$. (Use `Perpendicular Plane` tool).
     - Plot a point D on the plane. Using `Prism` tool click on the parallelogram and on D to complete the parallelepiped.
     - Find the volume using `Volume` tool. Change the position of D and observe the volume.
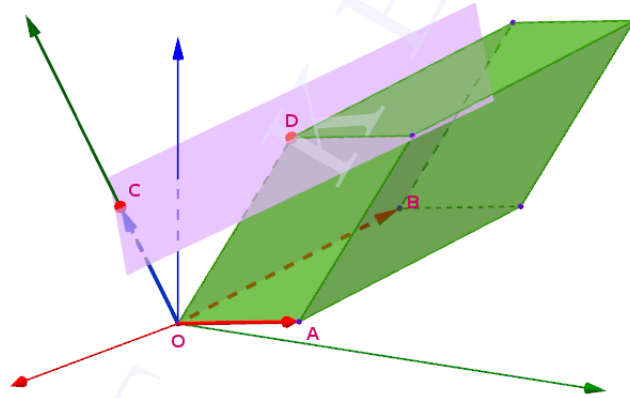
Additional Activities

Activity 38.A Cauchy Schwarz Inequality

In this activity we explore the Cauchy Schwarz Inequality which says that for two vectors $\vec{A}$ and $\vec{B}$

$$|\vec{A} \cdot \vec{B}| \leq |\vec{A}||\vec{B}|$$

Let $\vec{A} = a\hat{i} + b\hat{j}$ and $\vec{B} = b\hat{i} + a\hat{j}$.
Then the dot product of these vectors will be

$$\vec{A} \cdot \vec{B} = ab + ab = 2ab$$

The modulus of both the vectors

$$|\vec{A}| = |\vec{B}| = \sqrt{a^2 + b^2}$$

In this case the Cauchy Schwarz Inequality can be expressed as

$$2ab \leq a^2 + b^2$$

procedure

- Create two sliders **a** and **b** with minimum value of 0 and maximum value of 10.

- Create the points $A(0,0), B(a,b), C(b,a)$

- Construct vectors from A to B and A to C

- Create a square with one side as the length of the vector v. This can be done by using the Regular Polygon tool with the points A and C and 4 sides

- Create points $(b,0)$, $(0,2a)$ and $(b,2a)$. The points created are F, G and H respectively.

- Create a polygon AFHG

    Move sliders **a** and **b** and observe the area of the two polygons. Which one is greater?

    Can you theoretically argue why this is so?

    Move the sliders a and b and find a situation when the areas of the two polygons become equal. What do you observe regarding the value of both a and b. Can you prove this theoretically.

Lab 39

Line in space

Aim

- To derive equation of straight line in space

- To derive a method to find shortest distance between skew lines

Concepts

- Line in space

- Concept of vectors and vector operations

- Skew lines

Discussion

We derive the equation of a line passing through a given point and parallel to a given vector using GeoGebra. We discuss straight lines satisfying different conditions and construct them. With the help of an applet we derive a method to find the shortest distance between skew lines

Activity 39.1 Equation of a line

Procedure

- Open 3D graphics. Plot the points $O(0, 0, 0)$, $A(3, 4, 5)$

- Construct the vector $\vec{a} = \overrightarrow{OA}$ (input command `a=Vector(O,A)`) and $\vec{b} = 2\hat{i} + \hat{j} + \hat{k}$ (input command `b=Vector((2,1,1))`)

- Create a number slider λ with increment 0.01 and create a vector $\vec{u} = \lambda \vec{b}$ (input `u=λ*b`)

- Using `Vector from Point` tool click on the point A and on $\vec{u}$ to create the vector $\vec{v}$ equal to $\lambda \vec{b}$ with initial point A. Rename the end point of $\vec{v}$ as P. (If the end point is not visible use the input command `Point(A,u)`). Show the coordinates of P

- Hide $\vec{u}$

- Trace the point P. Animate the slider λ and observe the path of P. Use `Locus` tool to get the path of P

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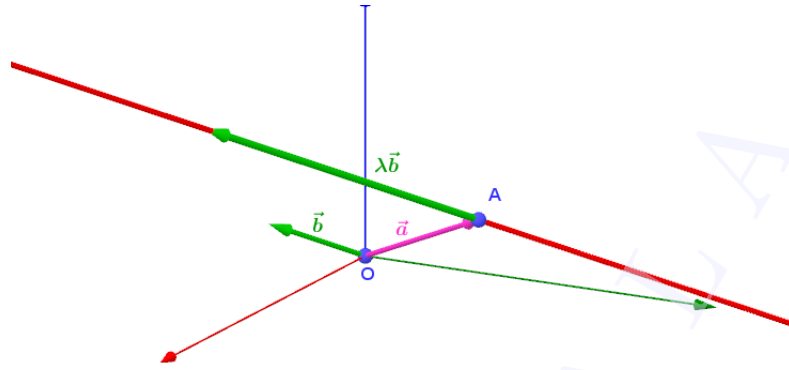
📖 What is the direction of the line obtained as the path of P ? Find the coordinates of a point (other than A) on this line.

📖 Write the vector $\overrightarrow{AP}$ when $\lambda = 2$. What is the distance of P from A. Verify your answer using distance tool

📖 Find any two points on the line which are equidistant from A.

📖 Find a point on the line which is at a distance of 5 units from A

📖 If $\vec{r}$ is the position vector of the point P , Write $\vec{r}$ in terms of $\vec{a}$ and $\vec{b}$

Activity 39.2 Construction of lines

We discuss the construction of lines passing through a given point and parallel to a given vector

Procedure

- Plot the point $A(2, 3, 2)$ and construct the vector $\vec{b} = \hat{i} + 2\hat{j} + 2\hat{k}$.

- Construct the line passing through A and parallel to $\vec{b}$. (Input command `Line(A,b)`)

- Create Input Boxes for A and $\vec{b}$.

📖 Construct the lines according to the given conditions by selecting suitable A and $\vec{b}$. Complete the following table.

Sl.No.	Conditions	A	$\vec{b}$
1	Along x axis		
2	Parallel to y axis but doesn't coincide with it		
3	Parallel to x axis, at a distance of 4 units from it		
4	Parallel to xy plane at a distance of 3 units from it		
5	Parallel to y axis, at a distance of 4 units from x axis		
6	Passing through $(1, 1, 1)$, direction ratios 2,3,-1		
7	Passing through $(1, 2, 3)$ and $\perp$ to $\hat{i} + 3\hat{j}$ and $2\hat{i} - \hat{j} + \hat{k}$		
8	Passing through the points $(2, 1, 1)$ and $(5, -3, 3)$		
9	Equation of the line $\dfrac{x - 2}{2} = \dfrac{y}{-2} = \dfrac{z + 1}{3}$		

Activity 39.3 Shortest Distance

We find the shortest distance between two skew lines geometrically.

Find the shortest distance between the lines

$\vec{r} = \hat{i} + \hat{j} + \lambda(2\hat{i} - \hat{j} + \hat{k})$ and

$\vec{r} = 2\hat{i} + \hat{j} - \hat{k} + \mu(3\hat{i} - 5\hat{j} + 2\hat{k})$

Procedure

- Construct the given lines (f and g). (Input command `f=Line((1,1,0), Vector((2,-1,1)))` gives the first line).

- Using the command `PerpendicularLine(f,g)` draw the line which is perpendicular to both of the lines.

- Find the points of intersection A and B of the perpendicular line with the given lines. Find the distance between them.

- Plot points C and D on the lines, draw the line segment CD and find its length.

- Change the position of C and D, observe its length and verify that it is minimum when CD coincides with AB.

  📖   Find the minimum length.

  Distance is minimum when DC is perpendicular to both of the lines. In the next activity we use this idea to derive an algebraic method to find the shortest distance between skew lines.

Activity 39.4 Shortest Distance - derivation

We derive a method to find the shortest distance between two skew lines

Procedure

Use Applet ML39.3

About the applet

- By moving the slider on the Graphics View we get descriptions and instructions one by one which helps the derivation of the shortest distance.

- In the 3D Graghics we get the corresponding pictures.

- Using `Rotate 3D Graphics View` tool to rotate the picture to view it from a convenient angle.

  📖   Move the slider, follow the descriptions and instructions, make necessary notes on the observation book and derive the method to find the shortest distance.

Lab 40

The Plane

Aim

- To derive the equation of a plane

- To explore the geometry of planes satisfying different conditions

Concepts

- Concept of plane

- Concept of vector and vector operations

Discussion

We derive the equation of a plane passing through a given point and perpendicular to a given vector. We also discuss some problems on planes and construct them

<u>**Tools and Commands**</u>

We discuss some tools and Input Commands for constructing planes

<u>**Commands**</u>

Sl.No	Input Command	Output Plane
1	Plane(Point,Plane)	Passes through the given point and parallel to the given plane
2	Plane(Point,Line)	Passes through the given point and contains the line
3	Plane(Line,Line)	Contains the lines (if they are intersecting)
4	Plane(Point,Point,Point)	Contains the given points
5	Plane(Point,Vector,Vector)	Contains the given point and parallel to the vectors
6	Plane(Polygon)	Contains the polygon
7	Plane(Conic)	Contains the given conic
8	PerpendicularPlane(Point,Line)	Contains the point and perpendicular to the line
9	PerpendicularPlane(Point,Vector)	Contains the point and perpendicular to the vector

Direct input of the equation gives the corresponding plane

<u>**Tools**</u>

Sl.No	Name of Tool	Output Plane
1	Plane through 3 Points	Same effect of Command 4
2	Plane	Same effect of Commands 2, 3, 4 and 6
3	Perpendicular Plane	Same effect of commands 8 and 9
4	Parallel Plane	Same effect of Command 1

Activity 40.1 Equation of a Plane

We derive the equation of plane passing through a given point and perpendicular to a given direction.

Procedure

- Plot the point $A(3,2,2)$ and construct the vector $\vec{a} = 2\hat{i} + \hat{j} + 5\hat{k}$ (input command `a=Vector((2,1,5))`)

- Construct the plane passing through A and perpendicular to $\vec{a}$

- Take any point B on the plane and draw the vector $\vec{u} = \overrightarrow{AB}$.

 📖   Find $\vec{a} \cdot \vec{u}$ at different position of B. Why it happens so?

  📖   Detach the point B from the plane (either use `Attach / Detach Point` tool or edit any one of the coordinates of B). Observe the change in $\vec{a} \cdot \vec{u}$. What happens to it when B is not on the plane? Why?

  📖   Write a necessary and sufficient condition for a point to be on the plane. Write the algebraic form of this condition. Hence find the equation of the plane.

  📖   Derive the equation of the plane passing through the point (x_1, y_1, z_1) and perpendicular to the vector $a\hat{i} + b\hat{j} + c\hat{k}$.
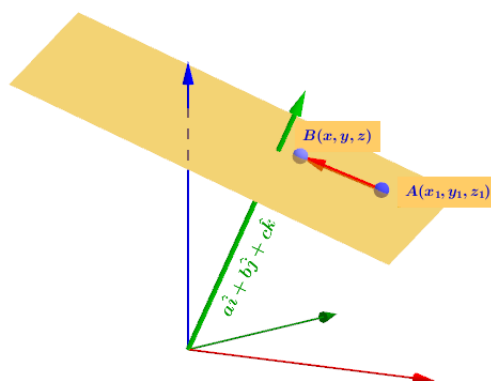
Activity 40.2 Construction of Planes

📖   Construct the planes as per the conditions given in the following problems. You can use any tool to construct vectors and lines but use only the `Perpendicular Plane` tool to construct the plane. Note the method of construction on your observation book. You can verify the result using any direct tool.

1. Plane passes through the point $(2,2,3)$ and parallel to the vectors $3\hat{i} - 2\hat{j} + \hat{k}$ and $\hat{i} + 3\hat{j}$.

2. Plane passes through the point $(1,-2,3), (1,1,4)$ and $(-3,2,2)$

3. Plane contains the line $\vec{r} = \hat{i} + 2\hat{j} - 3\hat{k} + \lambda(3\hat{j} + \hat{k})$ and parallel to $\vec{r} = 3\hat{i} + 2\hat{j} + \mu(2\hat{i} + \hat{j} + \hat{k})$

4. Plane contains the lines $\vec{r} = \hat{i} - \hat{j} - \hat{k} + \lambda(\hat{i} + 3\hat{j})$ and $\vec{r} = 3\hat{i} - 5\hat{j} - \hat{k} + \mu(\hat{i} + 3\hat{j})$

5. Plane perpendicular to the vector $2\hat{i} - 2\hat{j} + 2\hat{k}$ and at a distance of 5 units from the origin.

6. Plane passes through the point $(3, 2, 4)$ and parallel to the plane $x + y + 2z = 2$

7. Plane passes through the point $(1, 2, 3)$ and perpendicular to the planes $2x + y + 2z = 3$ and $x - 3y + z = 0$

8. Plane at a distance of 4 units from the plane $3x + 2y - z + 3 = 0$
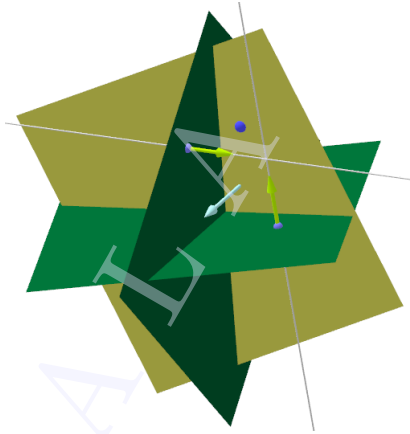
Lab 41

Intersection of Planes

Aim

- To explore the geometry of a linear equation of the form $a_1x+b_1y+c_1z+k(a_2x+b_2y+c_2z)=0$

- To study different cases of intersection of three planes with the help of matrices.

Concepts

- Plane and its equation

- Solution of a system of linear equations

- Consistency of a system of equations

Discussion

We discuss the formation of equation of a plane passing through the line of intersection of given planes. We compare the number of solutions of a sysem of linear equations in three variables with different positions of three planes.

Activity 41.1 Plane passing through the intersection of planes

Procedure
- Construct the planes $x+y+z=3$ and $x+y-z=1$.
- Create a number slider **a**.
- Give the input `x+y+z-3+a(x+y-z-1)=0` to get a third plane.

  📖 Change the value of **a** and observe the third plane. What peculiarity do you observe?

  📖 Find the value of **a** for which the third plane coincides with the first plane.

  📖 For what value of **a** does it coincides with the second plane?

- Edit the equation of the first two planes so that they are parallel. Also edit the equation of the third plane accordingly.

  📖 Change the value of **a** and observe the third plane. What peculiarity do you observe?

  📖 Find the value of **a** for which the third plane coincides with the first plane

  📖 For what value of **a** does it coincides with the second plane?

Activity 41.2 System of Linear Equations-Visualisation

A linear equation of the form $ax + by + cz + d = 0$ represents a plane in space. Geometrically, a solution of a system of linear equations is a point of intersection of a set of planes. We discuss different cases - unique solution, infinite solutions and no solution - geometrically. We also use matrices for the discussion.

Use Applet ML41.2

About the applet

- On Graphics View we can see three equations in x, y and z and the steps of solution. Clicking on the 'SOLUTION' button we get the solution of the equations.

- By clicking on the equations we can edit them.

- On 3D Graphics the corresponding planes and line of intersection of each pair of planes are shown.

- We can use Check Boxes to show/hide the lines of intersection.

- On Graphics View Yellow coloured Text Boxes are given corresponding to each step of solution. Clicking on it we can see the corresponding change in the planes and their line of intersection on 3D Graphics.

- While clicking on some of the Yellow coloured Text Boxes a Green coloured bullet will appear, clicking on which we can show/hide the previous position of the plane.

- The button 'DISCUSSION POINTS' gives the points to be kept in mind while learning with the help of the applet.

- The button 'HELP' gives a small description about the operation of the applet.

- The button 'GIVEN EQUATIONS' leads to the initial position of the planes.

Procedure

Let $AX = B$ represents a system of linear equations. If same elementary row transformations are done on A and B until A becomes the identity matrix, then B gives the solution of the system of equations.

Geometrically the point of intersection of the corresponding planes gives the solution of the equations. Even though the plane changes at each step of the row transformation we get the solution at the final stage. In the following activity observe the changes in the planes and try to find the reason of this.

- Using the applet find the solution of the system of equations $2x - y + 3z = 6$, $3x + 5y = 1$ and $-5x + 2y + 2z = 3$.

  📖 Observe the changes in the planes. Write the reason for the fact that the point of intersection remains unaltered even though the planes change.

  📖 What happens to the planes at the final stage?

Activity 41.3 Number of solutions - Visualisation

📖 Use the applet ML41.3, edit the equations so that the planes satisfy the given conditions. In each case, find the number of solutions geometrically. Also find $|A|$ and (adj $A)B$. Make a comparison between the number of solutions and the value of $|A|$ and (adj $A)B$ (You may use the applet ML41.3 to find adjoint of matrix). Write the equations in your observation book.

1. Planes are parallel and non coinciding.

2. Two planes are parallel and non coinciding and the third plane is not parallel to them.

3. Two planes are coinciding and the third plane is not parallel to them.

4. One of the equations is a linear combination of others.

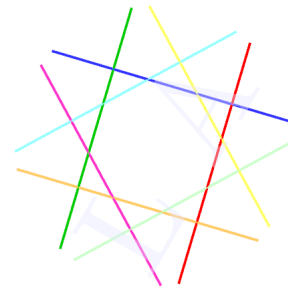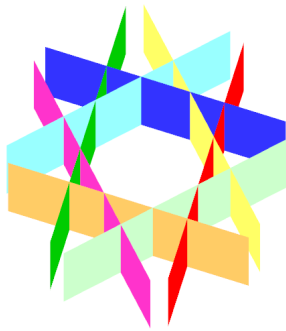5. Planes intersects pairwise and the three lines of intersection are parallel to each other.
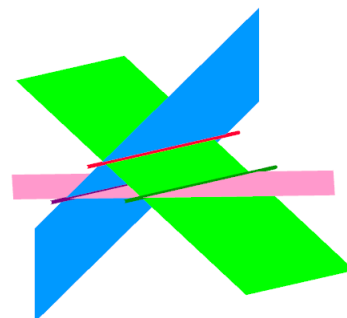
Activity 41.4 Construction of Planes

Construct the following objects given as in the following pictures. Cross section of each object is also given.

1
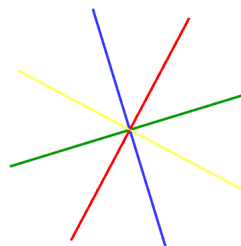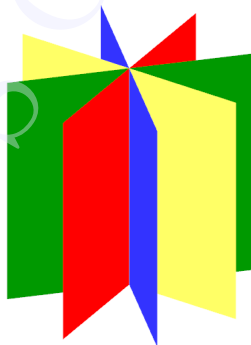
2

👉

An equation of the form $ax + by = c$ gives a plane perpendicular to the xy plane. The 2D equation of the line of intersection of the given plane with the xy plane is $ax + by = c$.

Lab 42

Linear Programming Problem

Aim

- To explore the geometrical solution of linear programming problem

- To understand the difference between feasible region, infeasible region, bounded and unbounded regions.

Concepts

- Corner point method for solving linear programming problem

- Solution of LPP having unbounded feasible region

Discussion

In this lab we do activities related to the linear programming problem which is to find the optimum value of an objective function subjected to a set of constraints. The function whose maximum or minimum is to be found is the objective function. The rules governing the values of variables and relations between variables are the contraints which are normally provided as inequalities. The corner point method is utilized to find the geometric solution of such set of equations.

Activity 42.1 Corner Points and Solution of a Linear Programming Problem

Consider the Linear Programming Problem
Maximise $z = x + 2y$
subject to the constrains

$$y - x \leq 1$$
$$x + 6y \leq 15$$
$$4x - y \leq 10$$
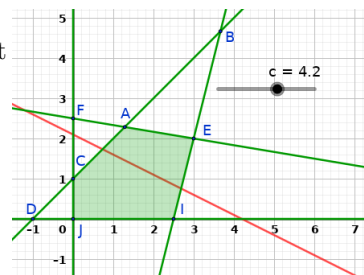$$x \geq 0, y \geq 0$$

procedure

- We can construct the feasible region using the input command
  $y - x <= 1 \&\& x + 6y <= 15 \&\& 4x - y <= 10 \&\& x >= 0 \&\& y >= 0$.
  A region named **a** is created.

- In the input bar, type the command `Corner(a)` to get the corner points.

  📖 Identify the corner points which are part of the feasible region.

  📖 Fill the following table

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Corner Point	x Coordinate	y Coordinate	$z = x + 2y$

📘 Which corner point of the polygon has the largest value of $z = x + 2y$?

📘 Which corner point has the minimum value of $z = x + 2y$?

Is there any other point inside the feasible region which can have a greater value of z than that at the corner points. To check this we do the following steps.

• Create a slider **c** with Min : -1, Max : 10 and Increment : 0.1

• Draw the line $x + 2y = c$

• Set the value of **c** at 1 and observe whether there is any points in the feasible region satisfying the condition $x + 2y = c$

• Gradually increase the value of **c** until there exist atleast one point of the feasible reagion is on the line $x + 2y = c$. Identify that point and find the value of c

• What is the maximum value of z in the feasible region.

Activity 42.2 Infinite Solutions

In this activity we discuss a linear programming problem having an infinite number of solutions.

Consider the Linear Programming Problem
Maximise $z = 3x + 9y$
subject to the constrains

$$x + 3y \leq 60$$
$$x + y \geq 10$$
$$x \leq y$$
$$x \geq 0, y \geq 0$$

Procedure

• Construct the feasible region using the input command
$x + 3y <= 60\&\&x + y >= 10\&\&x <= y\&\&x >= 0\&\&y >= 0.$

• Find the corner points using the input command `Corner(a)`

• Identify the corner points of the feasible region

• Find the values of z at the corner points

📘 Note down the maximum value and the points where this maximum value appears.

📘 Does the maximum occur at a single point or multiple points ?

Is there any other point inside the feasible region at which z is maximum ?
To check this we do the following steps.

- Create a slider **c** with Min : 0, Max : 200 and Increment : 0.1

- Draw the line $3x + 9y = c$

- Increase the value of **c** until there is atleast one point of the feasible reagion is on line.

  📖   What is the maximum value of **c** and what happens to the line at this stage

  📖   What does it infer about the points at which z is maximum ?

  📖   Is there any relation between the slope of the objective function with the slope of any of the lines obtained from the constrains ?

Activity 42.3 Unbounded feasible regions

Consider the linear programming problem

Determine graphically the minimum and maximum values of the objective function $z = 5x - 2y$ subject to the constraints:

$$2x{-}y \geq {-}5$$
$$3x + y \geq 3$$
$$2x{-}3y \leq 12$$
$$x \geq 0, y \geq 0$$

procedure

- Construct the feasible region.

- Plot the corner points of the feasible region.

  📖   Find the values of the objective function at the corner points

  📖   Identify the corner points at which z is maximum or minimum and find the values.

- Is there any other point inside the feasible region at which z is maximum or minimum ? We can check this as follows.

- Create a slider **c** with Min : -15, Max : 100 and Increment : 0.5

- Draw the line $5x - 2y = c$

- Change the value of **c** such that the line passes through the corner point which gives the maximum value.

- Increase the value of **c** further and observe the portion of the line inside the feasible region.

  📖   Is there exist any higher value of **c** for which the line lies completely ouside the feasible region.

  📖   What does it infer about the maximum value of the objective function in the feasible region.

  📖   What about the minimum value of the objective function in the feasible region.

Activity 42.4 Infeasible Solutions

There are cases where we cannot get a feasible region. Consider the following LPP.

Minimise $z = 3x + 2y$

subject to the constraints :

$$x + y \geq 8$$
$$3x + 5y \leq 15$$
$$x \geq 0, y \geq 0$$

What do you observe about the feasible region? Why?

Lab 43

Probability

Aim

To explore geometrically the concept of Probability, Conditional Probability and Bayes Theorem

Concepts

Probability, Conditional Probability, Bayes Theorem

Discussion

Probability is a number denoting the likelihood of an event to happen. It is expressed as a number between 0 and 1 in which case 0 will denote that the event is not going to occur and 1 will denote that the event will surely occur. This number is sometimes represented as a percentage ranging. In this lab we explore probability from a geometrical perspective.
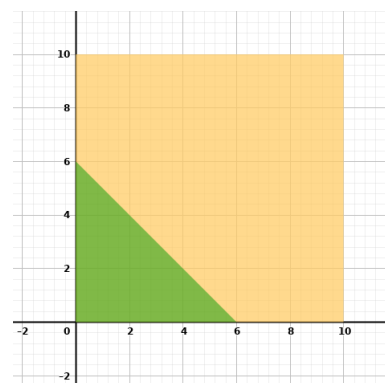
Activity 43.1 Probability as ratio of Areas

1. Suppose that two numbers whose values are between 0 and 10 are picked randomly. What is the probability that the sum of these numbers is less than 6?

   Procedure

   Let us explore this problem through geometry.

   - Let the two numbers picked be x and y. Then the given conditions are
     $0 < x < 10$ and $0 < y < 10$.The sample space in our case is all the numbers that satisfy these conditions. When viewed geometrically, this is any point inside a square of side 10. Draw the square with corner points A(0,0), B(10,0), C(10,10) and D(0,10)
   - The sum of these numbers less than 6 can be expressed as $x + y < 6, x > 0$ and $y > 0$. draw the line $x + y = 6$ and identify the triangular region representing $x + y < 6$, in side the square.

   - Required probability $= \dfrac{Area\ of\ the\ triangle}{Area\ of\ the\ rectangle}$

   📖   Find the probability.

   Note :- You can draw the regions directly by giving input commands.

   `0<x<10&&0<y<10` gives the rectangular region and

   `x+y<6&&x>0&&y>0` gives the triangular region.

2. If two numbers, whose values are between -6 and 6 are picked randomly. What is the probability that the sum of their squares is less than 25?

3. Suppose that two numbers whose values are between 0 and 10 are picked randomly. What is the probability that the product of these numbers is less than 25?
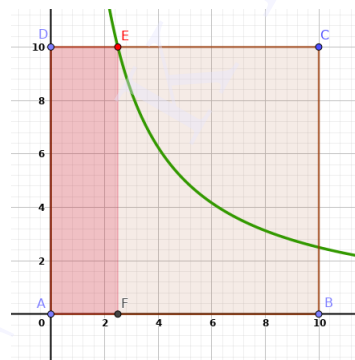
   Procedure

   - Here also the sample space is the square represented by $0 < x < 10$ and $0 < y < 10$. Draw this square (ABCD) using Polygon tool.

   - Product of numbers is less than 25 can be expressed as $xy < 25$. We can write it as
     $$y < \frac{25}{x}$$

   - Let us draw the curve $y = \frac{25}{x}$ using the input command `f(x)=25/x`

   - The area under this curve and inside the square is the desired space and hence
     $$\text{Required probability} = \frac{\textit{Area of the part of the square below the curve}}{\textit{Area of the square}}$$

   - Plot the point of intersection E with the top edge of the square.

   - The area under this curve inside the square consists of two parts as shown in the figure. A rectangular area and the area between the curve and the x axis.

   - Width of the rectangular region is the x coordinate of E. So the area of the rectangular region is obtained by giving the input command $10x(E)$.

   - To get the area under the curve we can use definite integrals. That is by using the command `Integral(f,x(E),10)`

   📖 Evaluate the required probability.

4. Let us have an extension to above problem.

   Suppose that two numbers whose values are between 0 and 10 are picked randomly. What is the probability that their sum is greater than 5 and the product is less than 25?

Activity 43.2 Conditional Probability

1. In the figure there is a rectangle and a circle inside a square. A point is marked inside the square,

   1. What is the probability that the point will be inside the rectangle

   2. What is the probability that the point will be inside the circle

   3. If it is known that the point is inside the circle, what is the conditional probability that it is inside the rectangle ?

   4. If it is known that the point is inside the rectangle, what is the conditional probability that it is inside the circle ?
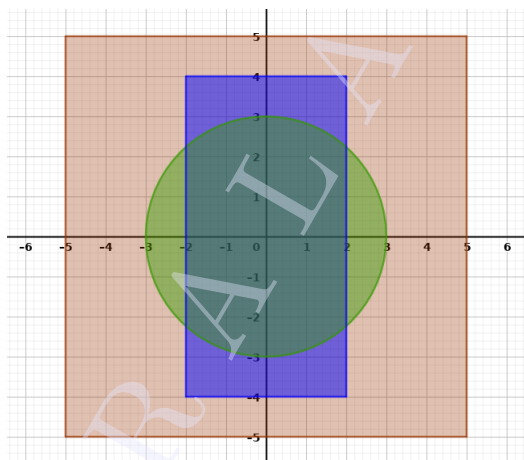
First two problems are same as that we discussed in the last activities. Now let us discuss the third problem.

Procedure

- Let us name the events as

  E : The point is inside the rectangle.

  F : The point is inside the circle.

  So we are asked to find $P(E|F)$.

- We know that $P(E|F) = \dfrac{P(E \cap F)}{P(F)}$

  We can find it in terms of the area of the regions.

- $P(E|F) = \dfrac{\textit{Area of the region common to the rectangle and the circle}}{\textit{Area of the circle}}$

- Equation of the circle is $x^2 + y^2 = 9$. We can find the common area using definite integrals.

- Area of the common region above the x axis is given by the input command
  `Integral(sqrt(9-x^2),-2,2)`

  📖   Find the area of the common region, area of the circle and hence find $P(E|F)$

  📖   Find the conditional probability that the point is inside the circle, if it is known that it is inside the rectangle.

2. Two numbers, between -5 and 5 are picked randomly. If it is known that their sum is less than 5 what is the probability that the sum of their squares is less than 25

Additional Activities

Activity 43.A A triangle problem

Suppose we have a stick of unit length which is broken into three pieces by cutting at two random points. What is the probability that the three pieces can be joined end to end to form a triangle?

First let us formulate the problem. Let us take the length of one piece as x and the length of the second piece as y. The third piece will have a length $1 - x - y$. To get hold of our universe, we first concentrate on the constraints of the three pieces. The first piece will have the restriction that

$$0 < x < 1$$

The second piece will have the constraint that

$$0 < y < 1$$

. The third piece will have the constraint that

$$0 < 1 - x - y < 1$$

Procedure

- Enter the expression $0 < x < 1$ in the input bar. A region is created.

- Enter the expression $0 < y < 1$ in the input bar. A region is created.

- Enter the expression $0 < 1 - x - y < 1$ in the input bar. A region is created. From the figure below it is clear that the intersection of the three regions is the triangle bounded by the vertices at the origin, (1,0) and (0,1)
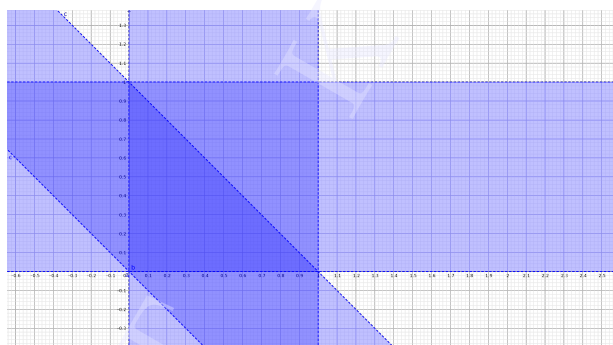
- Hide the regions.

- Create three points A(0,0), B(1,0) and C(0,1)

- 📖 Using polygon tool, create a triangle ABC. The triangle $q1$ is created. This is the universe. Observe the area.

When a triangle is formed from these three pieces, the triangle inequality should hold. The sum of any pairs of sides ahould be greater than the third side. So if y and $1 - x - y$ are taken together then

$$y + 1 - x - y > x \Rightarrow 1 - x > x \Rightarrow 1 > 2x \Rightarrow x < \frac{1}{2}$$

. Similar arguments can be used for the other sides as well to get

$$0 < x < \frac{1}{2}$$

$$0 < y < \frac{1}{2}$$

$$0 < 1 - x - y < \frac{1}{2}$$

- Enter the expression $0 < x < \frac{1}{2}$ in the input bar. A region is created.

- Enter the expression $0 < y < \dfrac{1}{2}$ in the input bar. A region is created.

- Enter the expression $0 < 1 - x - y < \dfrac{1}{2}$ in the input bar. A region is created. From the figure below it is clear that the intersection of the three regions is the triangle bounded by the vertices at $(\frac{1}{2},0)$, $(\frac{1}{2},\frac{1}{2})$ and $(0,\frac{1}{2})$
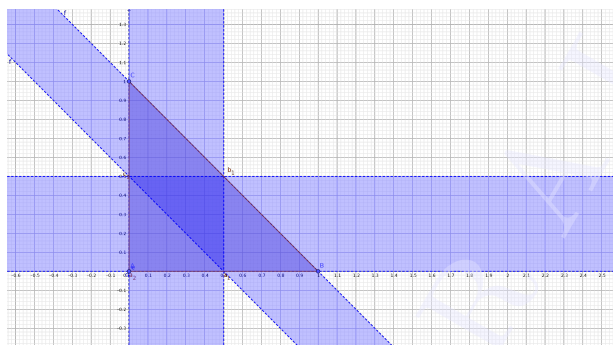
- Hide the regions.

- Create three points $D(\frac{1}{2},0)$, $E(\frac{1}{2},\frac{1}{2})$ and $F(0,\frac{1}{2})$

- 📖   Using Polygon tool, create a triangle DEF. The triangle $q2$ is created. This is the desired area. Observe the area.

- 📖   Calculate the probability that the three pieces can be joined end to end to form a triangle.

Activity 43.B Obtuse triangles

In this activity we build up on the previous activity. Use the same applet developed in the previous activity. Here we explore the probability that the triangles formed as in the previous activity are obtuse. Pythagoras stated his theorem for right angles as the sum of squares of two sides should be the square of the hypotenuse. For a triangle if square of any side is greater than the sum of squares of other sides, it would be obtuse. This could happen if

$$x^2 > y^2 + (1 - x - y)^2$$

or

$$y^2 > x^2 + (1 - x - y)^2$$

or

$$(1 - x - y)^2 > x^2 + y^2$$

These conditions will simplify to

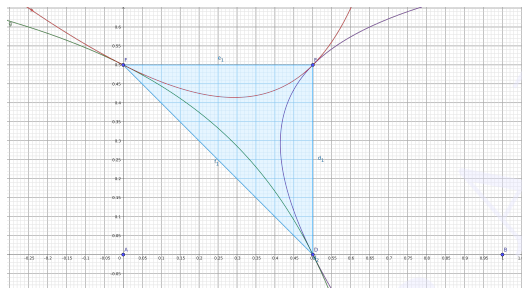$$y < 1 - \frac{1}{2(1 - x)}$$

$$y > -x + \frac{1}{2(1 - x)}$$

and

$$x > -y + \frac{1}{2(1 - y)}$$

Procedure

- Hide the polygon $q1$ from the previous applet.

- The triangle $q2$ is now our universe.

- In the input bar enter the expression $y = 1 - 1/(2(1-x))$. The curve g is created.

- In the input bar enter the expression $y = -x + 1/(2(1-x))$. The curve h is created.

- In the input bar enter the expression $x = -y - 1/(2(1-y))$. The equation $eq1$ is created. The result is shown in Figure below.

- 📖   Use the command to find the area between the curve g and the hypotenuse of the triangle, say f_1. Enter the command $IntegralBetween(g, f_1, 0, 0.5)$. The values of 0 and 0.5 specify the start and end points of the integration. Observe the area.

- 📖   Use the command to find the area between the curve h and the top side of the triangle, say e_1. Enter the command $IntegralBetween(e_1, h, 0, 0.5)$. The values of 0 and 0.5 specify the start and end points of the integration. Observe the area. The result is shown in the following Figure.
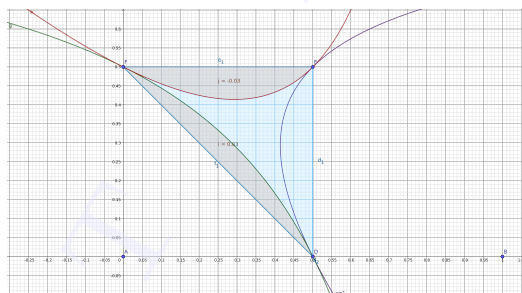
👉
If the area obtained by using the IntegralBetween command is negative, reverse the order of the two curves. The command is order sensitive.

- To find the area of the third portion, we use the argument of symmetry. It can be seen that both the areas of the other two pieces are same. The expression of the second and the last curve is similar, except for the change in variables. Hence it can be safely argued that the third curve will also have a similar area to the others.

   📖   Compute the probability that the formed triangle is obtuse.